Quiz Section 3

Review

1) **Chain Rule.** \( P(\mathcal{A} \cap \mathcal{B}) = \ldots \).

2) **Law of Total Probability.** If the events \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are a partition of \( \Omega \), then for any event \( \mathcal{B} \),
\[
P(\mathcal{B}) = \ldots.
\]

3) **Bayes Rule.** For any events \( \mathcal{A} \) and \( \mathcal{B} \),
\[
P(\mathcal{A} | \mathcal{B}) = \ldots.
\]

4) **Union Bound.** For events \( \mathcal{A}_1, \ldots, \mathcal{A}_n \),
\[
P\left( \bigcup_{i=1}^{n} \mathcal{A}_i \right) \leq \ldots.
\]

5) **Pairwise Independence of Events.** The events \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are ...
   (a) ... independent, if \( \ldots \).
   (b) ... pairwise independent, if \( \ldots \).

**Task 1 – Sequential Processes**

We revisit the following question from Section 2: We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > \cdots > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

We are interested in the following question: What is the probability the first card on the deck is (strictly) larger than the second one?

Here, however, we want to find a different solution.

a) Towards computing the probability of the above event, model the underlying probability space as a two-layered graph as described in class. Use as few nodes as necessary on each layer.

   We do not draw the whole graph here. Think of the card values as being the integers \( \{2, 3, \ldots, 14\} \). But on the first layer, the 13 card values are equally likely (we ignore the suit) – i.e., we get each card value with probability \( 1/13 \). On the second layer, we have only two nodes: “Not Lower” and “Lower”, representing whether the second card has lower value or not. Note that for a card of value \( i \in \{2, 3, \ldots, 14\} \), there are \( 4(i - 2) \) cards of lower values, and \( 51 \) cards remaining on deck. Thus, the probability of the second card being lower than the first one, if the first card has value \( i \), is \( 4(i - 2)/51 \), and this gives us the probabilities on the edges.

b) Use a) to give a direct calculation of the probability.

   The probability of getting “lower” is now, by the law of total probability and the chain rule,
\[
\frac{1}{13} \cdot \sum_{i=2}^{14} (i - 2) \cdot \frac{4}{51} = \frac{4}{13 \cdot 51} \cdot \sum_{i=0}^{12} i = \frac{4}{13 \cdot 51} \cdot \frac{12 \cdot 13}{2} = \frac{24}{51} = \frac{8}{17}.
\]
c) Given the second card is lower than the first one, what is the probability the first card is an Ace?

We know that \( P(\text{"lower"} \cap \text{"Ace"}) = \frac{1}{13} \cdot \frac{4}{12} = \frac{4}{51} \). We also know that \( P(\text{"lower"}) = \frac{8}{17} \).

Therefore,

\[
P(\text{"Ace"}|\text{"lower"}) = \frac{P(\text{"lower"} \cap \text{"Ace"})}{P(\text{"lower"})} = \frac{48}{13} \cdot \frac{17}{8} = \frac{6}{13} = \frac{2}{13}.
\]

**Task 2 – Infinite Processes**

Assume Alice is throwing a fair die with numbers 1, 2, \ldots, 6 on its sides. She keeps throwing it until she gets the number 1 or 3.

a) Let \( A_i \) be the event that Alice stops after \( i \) throws. Compute \( P(A_i) \) as a function of \( i \).

We can model this as a sequential process as in class. The probability of stop throwing the dice at each step is \( \frac{1}{3} \) to get a 1 or 3, and \( \frac{2}{3} \) to move to the next step. Therefore, the probability of stopping after \( i \) throws is

\[
P(A_i) = \left(\frac{1}{3}\right) \prod_{j=0}^{i-1} \left(\frac{2}{3}\right) = \frac{1}{3} \frac{1}{3-i+1} = \frac{1}{3} \frac{1}{3-i}.
\]

b) Verify that \( \sum_{i=1}^{\infty} P(A_i) = 1 \).

Note that

\[
\sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} \left(\frac{1}{3}\right) = \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = \frac{1}{3} \frac{1}{1-\frac{2}{3}} = \frac{1}{3} \cdot 3 = 1.
\]

**Task 3 – Coins**

There are three coins, \( C_1 \), \( C_2 \), and \( C_3 \). The probability of “heads” is 1 for \( C_1 \), 0 for \( C_2 \), and \( p \) for \( C_3 \). A coin is picked among these three uniformly at random, and then flipped a certain number of times.

a) What is the probability that the first \( n \) flips are tails?

We have

\[
\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (1-p)^n = \frac{1}{3} + \frac{1}{3} (1-p)^n.
\]

b) Given that the first \( n \) flips were tails, what is the probability that \( C_1 \) was flipped / \( C_2 \) was flipped / \( C_3 \) was flipped?

We use Bayes Rule, and obtain

\[
P(C_1 \mid n \text{ tails}) = \frac{\frac{1}{3} \cdot 0}{\frac{1}{3} + \frac{1}{3} (1-p)^n} = 0
\]

\[
P(C_2 \mid n \text{ tails}) = \frac{\frac{1}{3} \cdot 1}{\frac{1}{3} + \frac{1}{3} (1-p)^n} = \frac{1}{1 + (1-p)^n}
\]

\[
P(C_3 \mid n \text{ tails}) = \frac{\frac{1}{3} \cdot (1-p)^n}{\frac{1}{3} + \frac{1}{3} (1-p)^n} = \frac{(1-p)^n}{1 + (1-p)^n}
\]

**Task 4 – (Pairwise) Independence**

a) Use the law of total probability to show that if \( A \) and \( B \) are independent, then \( A \) and \( B^c \) are also independent.
We have, using the fact that $A$ and $B$ are independent and the law of total probability,

$$P(A \cap B^c) + P(A \cap B) = P(A) \cdot P(B) + P(A) \cdot P(A) \cdot P(A) = P(A).$$

Therefore,

$$P(A \cap B^c) = P(A) - P(A) \cdot P(B) = P(A)(1 - P(B)) = P(A) \cdot P(B^c).$$

b) Assume we draw two numbers $a, b$ from $\{0, 1, \ldots, N - 1\}$ uniformly and independently. Let $c = a + b \pmod{N}$. Let $A, B,$ and $C$ be the events that $a, b,$ and $c$ equals 0, respectively.

(i) Are $A, B,$ and $C$ independent?

No. While this can be proved more formally, it is clear that if $A$ and $B$ occurs, then $a = b = 0$, and thus $C$ also occurs. This contradicts independence.

(ii) Are $A, B,$ and $C$ pairwise independent?

Clearly, $A$ and $B$ are independent. For $C$, assume that $A$ happens, i.e., $a = 0$. Then, the only way for $C$ to occur is that $b = 0$, which happens with probability $1/N$. On the other hand, it is not hard to see that $P(C) = 1/N$, and thus the events are independent. The argument for $B$ and $C$ is symmetric.

Task 5 – Union Bound

a) Say that we choose a sequence of $n$ values from $[M]$. All sequences are equally likely. Let us consider the event that at least one of the values in the sequence is equal to 1. Use the union bound to show that for the probability of this event to be larger than $1/2$, we need $n > M/2$.

Let $A$ be the event that one of the $n$ values is 1. Let $A_i$ for $i \in [n]$ be the event that the $i$-th value is 1. Clearly, $A = \bigcup_{i=1}^{n} A_i$. Then,

$$P(A) = P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

We now need to compute $P(A_i)$. For this, note that there are $M^{n-1}$ sequences such that the $i$-th item is 1. In total, there are $M^n$ possible sequences, and thus $P(A_i) = 1/M$. Given this, we obtain that $P(A) \leq n/M$. Therefore, if $n \leq M/2$, we would have probability at most $1/2$ and thus $n > M/2$ must hold.

b) Solve a) assuming now that the $n$ elements in the sequence are distinct (and thus $n \leq M$).

The proof only changes locally. Now, $P(A_i) = 1/M$ still holds. This is because there are $\frac{M!}{(M-n)!}$ possible sequences, and among these, $\frac{(M-1)!}{(M-n)!}$ have a 1 in the $i$-th position. Thus,

$$P(A_i) = \frac{(M-1)!}{(M-n)!} \cdot \frac{1}{M!} = \frac{1}{M}.$$