CSE 312: Foundations of Computing II Autumn 2019

Quiz Section 3

Review

1) Chain Rule. $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) =$ ________.

2) Law of Total Probability. If the events $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are a partition of Ω, then for any event $\mathcal{B}$,

$$\mathbb{P}(\mathcal{B}) = \underline{\hspace{5cm}}\ .$$

3) Bayes Rule. For any events $\mathcal{A}$ and $\mathcal{B}$,

$$\mathbb{P}(\mathcal{A}|\mathcal{B}) = \underline{\hspace{5cm}}\ .$$

4) Union Bound. For events $\mathcal{A}_1, \ldots, \mathcal{A}_n$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right) \leqslant \underline{\hspace{6cm}}\ .$$

5) Pairwise Independence of Events. The events $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are ...

(a) ... independent, if ____.

(b) ... pairwise independent, if ____.

Task 1 – Sequential Processes

We revisit the following question from Section 2: We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > $\cdots$ > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

We are interested in the following question: What is the probability the first card on the deck is (strictly) larger than the second one?

Here, however, we want to find a different solution.

a) Towards computing the probability of the above event, model the underlying probability space as a two-layered graph as described in class. Use as few nodes as necessary on each layer.

b) Use **a)** to give a direct calculation of the probability.

c) Given the second card is lower than the first one, what is the probability the first card is an Ace?

Task 2 – Infinite Processes

Assume Alice is throwing a fair die with numbers $1, 2, \ldots, 6$ on its sides. She keeps throwing it until she gets the number 1 or 3.

a) Let $\mathcal{A}_i$ be the event that Alice stops after i throws. Compute $\mathbb{P}(\mathcal{A}_i)$ as a function of i.

b) Verify that $\sum_{i=1}^{\infty} \mathbb{P}(\mathcal{A}_i) = 1$.

Task 3 – Coins

There are three coins, C_1, C_2, and C_3. The probability of "heads" is 1 for C_1, 0 for C_2, and p for C_3. A coin is picked among these three uniformly at random, and then flipped a certain number of times.

a) What is the probability that the first n flips are tails?

b) Given that the first n flips were tails, what is the probability that C_1 was flipped / C_2 was flipped / C_3 was flipped?

Task 4 – (Pairwise) Independence

a) Use the law of total probability to show that if $\mathcal{A}$ and $\mathcal{B}$ are independent, then $\mathcal{A}$ and $\mathcal{B}^c$ are also independent.

b) Assume we draw two numbers a, b from $\{0, 1, \ldots, N-1\}$ uniformly and independently. Let $c = a + b \mod N$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the events that a, b, and c equals 0, respectively.

- **(i)** Are $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ independent?
- **(ii)** Are $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ pairwise independent?

Task 5 – Union Bound

a) Say that we choose a sequence of n values from $[M]$. All sequences are equally likely. Let us consider the event that at least one of the values in the sequence is equal to 1. Use the union bound to show that for the probability of this event to be larger than $1/2$, we need $n > M/2$.

b) Solve **a)** assuming now that the n elements in the sequence are distinct (and thus $n \leqslant M$).