Quiz Section 2

Review

1) Probability space. In a probability space $(\Omega, \mathbb{P})$, we have $\mathbb{P}(\omega)$ ______ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \mathbb{P}(\omega) =$ ______.

2) Disjoint events. The events $\mathcal{A}$ and $\mathcal{B}$ are *disjoint* if $\mathcal{A} \cap \mathcal{B} =$ __________

3) Additivity of Probability. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are mutually disjoint events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right) = \underline{\hspace{4cm}} .$$

4) Complement. For any event $\mathcal{A}$, $\mathbb{P}(\mathcal{A}^c) =$ ______________ .

5) Conditional Probability. $\mathbb{P}(\mathcal{B} \mid \mathcal{A}) =$ __________

6) Independent Events. Two events $\mathcal{A}, \mathcal{B}$ are **independent** if $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) =$ __________.
If $\mathbb{P}(\mathcal{A}) \neq 0$, this is equivalent to $\mathbb{P}(\mathcal{B} \mid \mathcal{A}) =$ __________.
If $\mathbb{P}(\mathcal{B}) \neq 0$, this is equivalent to $\mathbb{P}(\mathcal{A} \mid \mathcal{B}) =$ __________.

Task 1 –The Pigeonhole Principle

Show that in any group n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

a) Everyone has at least one friend.

b) At least one person has no friends.

Task 2 – Balls from an Urn – Take 1

Say an urn contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.

b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

c) What is the probability that at most one ball is red?

d) What is the probability that we get at least one green ball?

e) Repeat **c)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn.

Task 3 – Shuffling Cards

We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace $>$ King $>$ Queen $>$ Jack $>$ 10 $> \cdots >$ 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

Task 4 – Flipping Coins

A coin is tossed twice. The coin is "heads" one quarter of the time. You can assume that the second toss is independent of the first toss.

a) What is the probability that the second toss is "heads" given that the first toss is "tails"?

b) What is the probability that the second toss is "heads" given that at least one of the tosses is "tails"?

c) In the probability space of this task, give an example of two events that are disjoint but not independent.

d) In the probability space of this task, give an example of two events that are independent but not disjoint.

Task 5 – Balls from an Urn – Take 2

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

a) What is the probability that all three balls are all of the same color?

b) What is the probability that we get more than one red ball given the first ball is red?

Task 6 – Additivity of Probability

Use the additivity of probability to prove that

$$\mathbb{P}(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) = \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B}) + \mathbb{P}(\mathcal{C}) - \mathbb{P}(\mathcal{A} \cap \mathcal{B}) - \mathbb{P}(\mathcal{A} \cap \mathcal{C}) - \mathbb{P}(\mathcal{B} \cap \mathcal{C}) + \mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) \ .$$