Inclusion-Exclusion and Derangements

The goal here is to count the number derangements, i.e., the number of bijective (or one-to-one) functions\(^1\) \(\pi : [n] \to [n]\) which have the property that \(\pi(i) \neq i\) for all \(i \in [n]\). We are after a formula which is a function of \(n\) – call \(D(n)\) the number of such derangements. For example, \(D(3) = 2\). This is easy to see because the only two solutions are

\[(2, 3, 1)\]

and

\[(3, 1, 2)\]

(Here, \((a, b, c)\) is to be read as \(\pi(1) = a, \pi(2) = b,\) and \(\pi(3) = c\).) But what about \(D(n)\) for an arbitrary \(n\)? We are going to show that \(D(n)\) is close to \(n! - \frac{n!}{e}\), where \(e = 2.71828\ldots\) is Euler’s number.

**Setting up the problem.** Towards using the inclusion-exclusion principle, we need to think of the set of derangements as being described by the set of \(\pi\)’s satisfying

\[\pi(1) \neq 1 \land \pi(2) \neq 2 \land \pi(3) \neq 3 \land \cdots \land \pi(n) \neq n.\]

By DeMorgan’s laws, this is equivalent to counting the number of \(\pi\)’s that do not satisfy

\[\pi(1) = 1 \lor \pi(2) = 2 \lor \pi(3) = 3 \lor \cdots \lor \pi(n) = n.\]

We formalize this using sets. First of all, let \(S_n\) be the set of one-to-one \(\pi : [n] \to [n]\). We know that \(|S_n| = n!\) already. Also, for any \(i \in [n]\), let

\[A_i = \{\pi \in S_n : \pi(i) = i\}.\]

Note that we have defined \(n\) sets here, \(A_1, \ldots, A_n\). Now, by the above, we want the set of \(\pi\)’s which are in \(S_n\), but not in \(A_1 \cup \cdots \cup A_n\). In other words,

\[D(n) = |S_n \setminus (A_1 \cup \cdots \cup A_n)| = |S_n| - |A_1 \cup \cdots \cup A_n| = n! - |A_1 \cup \cdots \cup A_n|,\]

where the second equality follows from the fact that \(A_1 \cup \cdots \cup A_n \subseteq S_n\), because the \(A_i\)’s are defined as subsets of \(S_n\).

**Using inclusion-exclusion.** We now want to evaluate (1) above. By the inclusion-exclusion principle,

\[|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|.\]

To continue, we need to know what \(\bigcap_{i \in I} A_i\) is. Note that for any non-empty \(I \subseteq [n]\), having \(\pi \in \bigcap_{i \in I} A_i\) means that \(\pi(i) = i\) for all \(i \in I\), but there is no restriction on \(\pi(j)\) for \(j \in [n] \setminus I\).
therefore, there are \((n - |I|)!\) ways of choosing \(\pi(j)’s\ for \ j \in [n]\setminus I. \) (Remember, we look at \(\pi’\s\ which\ are\ one-to-one!) In other words,

\[
\left| \bigcap_{\pi \in \mathcal{P}} A_{\pi} \right| = (n - |I|)!
\]

for all non-empty \(I \subseteq [n].\)

Now, we plug (3) into (2), and obtain

\[
|A_1 \cup \cdots \cup A_n| = \sum_{\varnothing \neq I \subseteq [n]} (-1)^{|I|+1}(n - |I|)!
\]

Let us now simplify this. We are summing over all non-empty \(I \subseteq [n] – i.e., \ there\ are\ \(2^n - 1\) summands – and the summand is \((-1)^{|I|+1}(n - k)!\) for any \(I\ with\ size\ \(k,\ regardless\ of\ how\ \(I\ \) exactly looks like. Moreover, there are \(\binom{n}{k}\) such \(I’\s\ for each \(k \in [n].\) Therefore, we can instead write

\[
|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!(n-k)!} \cdot (n-k)! = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}.
\]

Note that the second equality follows from the definition of the binomial coefficient, and the last equality follows by simplifying \((n - k)!\) away.

To wrap up, let us plug (5) into (1), and obtain

\[
D(n) = n! - \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!} = n! + \sum_{k=1}^{n} (-1)^{k} \frac{n!}{k!} = \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!} = n! \sum_{k=0}^{n} (-1)^{k} / k!,
\]

where the first equality uses that \((-1)^{k+1} = (-1) \cdot (-1)^k\) (and this allows us to “flip” the sign!), and the second uses the fact that \((-1)^{k} \frac{n!}{k!} = 1\ for \ k = 0\ – so we have taken-in the \(n!\ as\ the \(k = 0\ term\ into\ the\ sum.

\textbf{Interpreting this.} We are actually done – i.e.,

\[
D(n) = n! \sum_{k=0}^{n} \frac{(-1)^{k}}{k!}
\]

is the best we can expect to prove as an \textit{exact} formula. However, we can get a feel of what this means, by observing that for any \(x,\)

\[
\sum_{k=0}^{n} \frac{x^k}{k!}
\]

converges to \(e^x\ as \(n \) goes to infinity. Therefore, in our case, \(\sum_{k=0}^{n} \frac{(-1)^{k}}{k!}\) goes towards \(e^{-1} = 1/e\ as \(n \) goes to infinity. Which explains why (as we claimed above) \(D(n) \approx n!/e\ as \(n \) grows large.

In fact, one can do even better, and show that \(D(n)\ is always the closest integer to \(n!/e\) (the latter is never an integer).
Was I supposed to come up with it by myself? This goes beyond the type of homework questions we are going to ask – but it is entirely solvable with the tools from class. The problem was first posed by Pierre Raymond de Montmort, a French mathematician, in 1708. He only solved it in 1713, and so did Nicholas Bernoulli, a Swiss mathematician, and one out of many Bernoullis who are famous mathematicians.