Homework 8

Due: Friday, December 6, by 11:59pm – no extensions accepted
Refer to the instructions on Homework 1

Task 1 – A Lazy Grader [8 pts]

Prof. Lazy decides to assign final grades in CSE 312 by ignoring all the work the students have done and instead using the following probabilistic method: each student independently will be assigned an A with probability $\theta$, a B with probability $3\theta$, a C with probability $\frac{1}{2}$, and an F with probability $\frac{1}{2} - 4\theta$. When the quarter is over, you discover that only 2 students got an A, 10 got a B, 60 got a C, and 40 got an F.

Find the maximum likelihood estimate for the parameter $\theta$ that Prof. Lazy used. Give an exact answer as a simplified fraction.

Task 2 – Maximum Likelihood Estimators [12 pts]

a) Let $x_1, x_2, \ldots, x_n$ be independent samples from a geometric distribution with unknown parameter $p$. What is the maximum likelihood estimator for $p$?

b) If the samples from the geometric distribution are 5, 4, 10, 2, 9, 6, 13, 9, what is the maximum likelihood estimator for $p$? Give an exact answer as a simplified fraction.

c) Let $x_1, x_2, \ldots, x_n$ be independent samples from an exponential distribution with unknown parameter $\lambda$. What is the maximum likelihood estimator for $\lambda$?

d) Suppose $x_1, x_2, \ldots, x_n$ are the numbers of independent requests arriving at a web server during each of $n$ successive minutes. Under the assumption that these numbers are independent samples from a Poisson distribution with unknown rate $\lambda$ per minute, what is the maximum likelihood estimator for $\lambda$?

e) Is your estimator from d) unbiased? Justify your answer.

Task 3 – Confidence Intervals [8 pts]

We have observed 9312 coin tosses, 4932 of which are heads. We assume that each coin toss is independently heads with probability $p$.

In the following, give all your answers to 4 significant digits.

a) What is the maximum likelihood estimate $\hat{p}$ of $p$ for the 9312 coin tosses?

   Hint: You do not need to re-derive the formulas obtained in class.

b) Approximate the 98% confidence interval for $p$. Give your answer as the least value of $\Delta$ such that $P(\hat{p} - \Delta \leq p \leq \hat{p} + \Delta) \geq 0.98$. You can assume that the variance of each coin flip (when interpreted as a Bernoulli 0/1 variable) is 0.25, and that the Central Limit Theorem applies.

Task 4 – Estimators for the Uniform Distribution [12 pts]

Let $x_1, x_2, \ldots, x_n$ be independent samples from the continuous uniform distribution on $[0, \theta]$, where $\theta$ is the unknown parameter.
a) What is the maximum likelihood estimator $\hat{\theta}$ for $\theta$?
\textbf{Hint:} It is perfectly acceptable to describe the behavior of the likelihood function (as a function of $\theta$) and infer the maximum from this description.

b) Compute the CDF $F_{\hat{\theta}}(x)$ for the random variable $\hat{\theta}$ obtained by applying your estimator to uniform random variables $X_1, \ldots, X_n$ which are uniform between 0 and $\theta$.
\textbf{Hint:} Focus first on the interval $0 \leq x \leq \theta$, but when you’re done with that, don’t forget to also define $F(x)$ on the rest of the real numbers.

c) From your answer to b) compute the probability density function $f_{\hat{\theta}}(x)$ of $\hat{\theta}$.

d) From your answer to c), compute $E(\hat{\theta})$. Is $\hat{\theta}$ an unbiased estimator of $\theta$?

e) Starting from the value of $E(\hat{\theta})$ you computed in d), determine an unbiased estimator of $\theta$ and explain why it is unbiased.
\textbf{Hint:} There are different unbiased estimators of $\theta$, but we want one using directly what you inferred in d).