Homework 8

Due: Friday, December 6, by 11:59pm – **no extensions accepted**
Refer to the instructions on Homework 1

Task 1 – A Lazy Grader [8 pts]

Prof. Lazy decides to assign final grades in CSE 312 by ignoring all the work the students have done and instead using the following probabilistic method: each student independently will be assigned an A with probability θ, a B with probability 3θ, a C with probability $\frac{1}{2}$, and an F with probability $\frac{1}{2} - 4\theta$. When the quarter is over, you discover that only 2 students got an A, 10 got a B, 60 got a C, and 40 got an F.

Find the maximum likelihood estimate for the parameter θ that Prof. Lazy used. Give an exact answer as a simplified fraction.

Task 2 – Maximum Likelihood Estimators [12 pts]

a) Let $x_1, x_2, \ldots, x_n$ be independent samples from a geometric distribution with unknown parameter p. What is the maximum likelihood estimator for p?

b) If the samples from the geometric distribution are 5, 4, 10, 2, 9, 5, 6, 13, 9, what is the maximum likelihood estimator for p? Give an exact answer as a simplified fraction.

c) Let $x_1, x_2, \ldots, x_n$ be independent samples from an exponential distribution with unknown parameter λ. What is the maximum likelihood estimator for λ?

d) Suppose $x_1, x_2, \ldots, x_n$ are the numbers of independent requests arriving at a web server during each of n successive minutes. Under the assumption that these numbers are independent samples from a Poisson distribution with unknown rate λ per minute, what is the maximum likelihood estimator for λ?

e) Is your estimator from **d)** unbiased? Justify your answer.

Task 3 – Confidence Intervals [8 pts]

We have observed 9312 coin tosses, 4932 of which are heads. We assume that each coin toss is independently heads with probability p.

In the following, give all your answers to 4 significant digits.

a) What is the maximum likelihood estimate $\hat{p}$ of p for the 9312 coin tosses?

Hint: You do not need to re-derive the formulas obtained in class.

b) Approximate the 98% confidence interval for p. Give your answer as the least value of Δ such that $\mathrm{P}(\hat{p} - \Delta \leqslant p \leqslant \hat{p} + \Delta) \geqslant 0.98$. You can assume that the variance of each coin flip (when interpreted as a Bernoulli 0/1 variable) is 0.25, and that the Central Limit Theorem applies.

Task 4 – Estimators for the Uniform Distribution [12 pts]

Let $x_1, x_2, \ldots, x_n$ be independent samples from the continuous uniform distribution on $[0, \theta]$, where θ is the unknown parameter.

a) What is the maximum likelihood estimator $\hat{\theta}$ for θ?

Hint: It is perfectly acceptable to describe the behavior of the likelihood function (as a function of θ) and infer the maximum from this description.

b) Compute the CDF $F_{\hat{\theta}}(x)$ for the random variable $\hat{\theta}$ obtained by applying your estimator to uniform random variables $X_1, \ldots, X_n$ which are uniform between 0 and θ.

Hint: Focus first on the interval $0 \leqslant x \leqslant \theta$, but when you're done with that, don't forget to also define $F(x)$ on the rest of the real numbers.

c) From your answer to **b)** compute the probability density function $f_{\hat{\theta}}(x)$ of $\hat{\theta}$.

d) From your answer to **c)**, compute $\mathbb{E}\left(\hat{\theta}\right)$. Is $\hat{\theta}$ an unbiased estimator of θ?

e) Starting from the value of $\mathbb{E}\left(\hat{\theta}\right)$ you computed in **d)**, determine an unbiased estimator of θ and explain why it is unbiased.

Hint: There are different unbiased estimators of θ, but we want one using directly what you inferred in **d)**.