CSE 312: Foundations of Computing II Autumn 2019

Homework 7

Due: Wednesday, November 27, by 11:59pm.
Refer to the instructions on Homework 1

Task 1 – Probability Density Functions I [6 pts]

For two independent random variables X and Y, the PDF of $Z = X + Y$ is given by the convolution

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(y) \cdot f_Y(z-y)\, dy = \int_{-\infty}^{\infty} f_X(z-y) \cdot f_Y(y)\, dy \; .$$

Suppose X and Y are independent, and all are uniformly distributed real numbers between 0 and 1. Derive the pdf of $X + Y$.

Task 2 – Probability Density Functions II [6 pts]

Let X be a continuous random variable which follows the uniform distribution between 0 and 1. Derive the pdf of X^3.

Task 3 – Servers and Approximations [12 pts]

Server A sequentially handles 30 jobs, each of whose service times are i.i.d. (independent, identically distributed) with mean 50 milliseconds and standard deviation 10 milliseconds. Server B has an analogous workload, but its 30 jobs each have mean 52 milliseconds and standard deviation 15 milliseconds.

Express your answers in terms of Φ, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$. You do not need not to compute the numerical value, it is enough to express the probability in terms of Φ.

a) Estimate the probability that server A finishes in less than 1400 milliseconds. (Here, and below, approximate workloads as normally distributed.)

b) Estimate the probability that server B finishes in less than 1400 milliseconds.

c) Suppose that the two random variables $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, for $i \in \{1, 2\}$, are independent. What is $\mathbb{E}(X_1 - X_2)$? What is $\mathrm{Var}(X_1 - X_2)$?

d) Estimate the probability that server B finishes before A.

Hint: If X_1 and X_2 are independent normal random variables, how does $aX_1 + bX_2$ behave? (You can assume the behavior of approximately normal random variables.)

e) If you did part (d) correctly, you will discover that B has a non-negligible chance of finishing earlier than A, even though A has the smaller mean completion time. Explain how this is possible.

Task 4 – Properties of MGFs [8 pts]

X is a random variable with the moment generating function $M(t)$. Find random variables that are a function of X with the following moment generating functions:

a) $M(t) \cdot M(5t)$.

b) $e^{-10t} \cdot M(t)$.

Task 5 – More on MGFs [8 pts]

X is a random variable with moment generating function $M_X(t) = \frac{1}{(1-t)^2}$, and Y is a random variable with moment generating function $M_Y(t) = \frac{1}{(1-5t)^4}$. If $Z = X + Y$ and X, Y are independent, then

a) compute the moment generating function of Z.

b) compute the value of $\mathbb{E}\left(Z^2\right)$