CSE 312: Foundations of Computing II Autumn 2019

# Homework 6

Due: Wednesday, November 20, by 11:59pm.
Refer to the instructions on Homework 1

## Task 1 – Chernoff Bound [8 pts]

A certain city is experiencing a big crime wave. The city decides that it needs to put its police officers out into the streets to bring back security. The city is conveniently arranged into a $100 \times 100$ grid of streets. Each street intersection can be identified by two integers $(a, b)$ where $1 \leqslant a \leqslant 100$ and $1 \leqslant b \leqslant 100$. The city only has $1000$ police officers, so it decides to send each police officer to a uniformly random grid location, independent of each other. The city wants to make sure that every $20 \times 20$ subgrid (corresponding to grid points $(a, b)$ with $A \leqslant a \leqslant A + 19$ and $B \leqslant b \leqslant B + 19$ for valid $A, B$) gets more than $10$ police officers.

**a)** Use the Chernoff bound to compute the probability that a single subgrid gets at most $10$ police officers.

**b)** Use the union bound together with the result from above to calculate the probability that the city fails to meet its goal.

## Task 2 – Probability Density Functions [8 pts]

For this exercise, give exact answers as simplified fractions. Compute $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ if $X$ has probability density function given by ...

**a)** $f_X(x) = \begin{cases} c(1-x^6) & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

Determine the value of $c$ as part of your answer.

**b)** $f_X(x) = \begin{cases} c/x^5 & \text{if } x > 5 \\ 0 & \text{otherwise} \end{cases}$.

Determine the value of $c$ as part of your answer.

**Hint:** The variance does not necessarily need to be finite.

## Task 3 – Exponential Density [8 pts]

You throw a dart at a circular target of radius $r$. Let $X$ be the distance of your dart's hit from the center of the target. Your aim is such that $X$ follow the exponential density with parameter $4/r$.

**a)** As a function of $r$, determine the value $m$ such that $\mathbb{P}(X < m) = \mathbb{P}(X > m)$.

**b)** What is the probability that you miss the target completely?

## Task 4 – The Flea and the Beach Ball [8 pts]

A flea of negligible size is trapped in a large, spherical, inflated beach ball with radius $r$. At this moment, it is equally likely to be at any point within the ball. Let $X$ be the distance of the flea from the center of the ball. Using the fact that the volume of a sphere of radius $r$ is $4\pi r^3/3$, for $X$, find ...

**a)** ... the cumulative distribution function $F_X$.

**b)** ... the probability density function $f_X$.

**c)** ... the expected value $\mathbb{E}(X)$.

**d)** ... the variance $\mathrm{Var}(X)$.

## Task 5 – The Strange Die [8 pts]

Suppose you have a die that has probability $p$ of resulting in the outcome 6 when rolled, where $p$ is a continuous random variable that is uniformly distributed over $[0, \frac{1}{3}]$. Suppose you start rolling this die. (The value of $p$ does not change once you start rolling.) Give exact answers as simplified fractions.

**a)** Compute the probability that the first roll is 6.

**b)** Compute the probability that the first two rolls are both 6.

**c)** Let $\mathcal{S}_1$ be the event that the first roll is 6 and $\mathcal{S}_2$ be the event that the second roll is 6. Compute $\mathbb{P}(\mathcal{S}_2 \mid \mathcal{S}_1)$.

**d)** Are the outcomes of the first two rolls independent? Justify your answer.

**e)** Compute the probability that the first $k$ rolls are each 6. Compare this with the probability if it were a fair die.