CSE 312: Foundations of Computing II Autumn 2019

Homework 5

Due: Wednesday, November 13, by 11:59pm.
Refer to the instructions on Homework 1

Task 1 – Conditional Expectation [4 pts]

Assume we roll a six-sided fair die twice. Let X be the value of the first roll. Let $\mathcal{E}$ be the event that the sum of the two rolls is 6. Compute $\mathbb{E}(X \mid \mathcal{E})$.

Task 2 – Running Game [10 pts]

Bob's friends convince him to play the following game. In each round, he rolls a three-sided die. If he obtains 1 or 2, he needs to run for one or two minutes, respectively, around the house, and he moves to the next round. If he obtains 3, he runs for 3 minutes and the game ends. Let T be the number of minutes he is expected to run in total. (For example, if Bob gets twice a 1, and then a 3, then $T = 1 + 1 + 3 = 5$.)

a) Let $\mathcal{A}_3$ be the event that the first die roll is 3. What is $\mathbb{E}(T|\mathcal{A}_3)$?

b) Let $\mathcal{A}_1$ be the event that the first die roll is 1. Show that

$$\mathbb{E}(T|\mathcal{A}_1) = 1 + \mathbb{E}(T) \ .$$

Hint: Does the distribution of the number of minutes left to run in the game depend on how far Bob has already run?

c) Use the Law of Total Expectation to find $\mathbb{E}(T)$.

Task 3 – Poisson Distribution [8 pts]

The company Moonbound sells luxury spaceships. According to their historical sales, customers buy spaceships at an average rate of 5 per week. Spaceships are expensive for them to stock but, on the other hand, they hate to run out of spaceships and turn away a rich customer. Once a week, Moonbound restocks spaceships for the coming week. They want to have the minimum number of spaceships such that there is less than a 0.1% probability that they have more buying customers than spaceships during the next week. They cannot figure out how many spaceships this should be and turn to you for advice. After asking some questions, you determine that arrivals of paying customers seem to be independent of each other and decide that a Poisson distribution would be a good model.

Given this decision, how many spaceships should Moonbound stock for the coming week?

Task 4 – Markov's Inequality [6 pts]

The expected grade for a student in CSE 312 is 3.4 (out of 4.0). Use Markov's inequality to upper bound the probability that a student receives less than 2.8.

Task 5 – Pairwise Independence [12 pts]

Let $X_1, \ldots, X_n$ be random variables which are *pairwise independent*, i.e., for every $i \neq j$, the random variables X_i and X_j are independent. (Note that this *does not* mean that they are *mutually* independent!)

a) Show that

$$\mathbb{E}\left(\left(\sum_{i=1}^{n} X_i\right)^2\right) = \sum_{i=1}^{n} \mathbb{E}\left(X_i^2\right) + \sum_{i \neq j} \mathbb{E}\left(X_i\right) \cdot \mathbb{E}\left(X_j\right) \ .$$

Hint: Try this for $n = 3$ if the general answer is too hard. Make sure to only use the fact that the X_i's are pairwise independent.

b) Show that

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}\left(X_i\right) \ .$$

Hint: Use **a)** and the fact that $\mathrm{Var}\left(X\right) = \mathbb{E}\left(X^2\right) - \mathbb{E}\left(X\right)^2$.

c) Conclude that if $X_1, \ldots, X_n$ all have expectation μ and variance σ^2, then for $X = \sum_{i=1}^{n} X_i$,

$$\mathbb{P}\left(|X - n \cdot \mu| \geqslant c\right) \leqslant \frac{n\sigma^2}{c^2} \ .$$