Homework 4

Due: Wednesday, Oct 30, by 11:59pm.
Refer to the instructions on Homework 1

Task 1 – Random Variables [12 pts]

a) Let us define the random variable X such that $\mathbb{P}(X = 0) = 1/3$, $\mathbb{P}(X = 7) = 2/7$, and $\mathbb{P}(X = 11) = 8/21$. Compute $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

b) Anna, Brian, and Carl play the following game. Each of them puts \$3 on the table, and then secretly write a guess "heads" or "tails" in a sealed envelope, which they also put on the table. At the end, a fair coin is flipped, and the \$9 on the table are split among those who guessed the outcome correctly in their envelopes. Let W be Anna's net payoff (i.e., how much she takes home in excess of the \$3 she puts on the table).

Compute $\mathbb{E}(W)$ and $\mathrm{Var}(W)$, assuming that all three players put a random and indepent guess in their envelopes, i.e., they are each equally likely to guess heads or tails.

c) Assume that we roll a four-sided dice twice, i.e., the sides take values $\{1, 2, 3, 4\}$. The two rolls here are independent. Let A be the random variable describing the number of rolls with even outcome. Let B be the random variable describing the number of rolls with odd outcome.

- Give the joint PMF of A and B.
- Compute $\mathbb{P}(A \geqslant B)$.
- Compute $\mathbb{P}(A = 1 \mid B \geqslant 1)$.

Task 2 – Multi-Processor Systems [12 pts]

We look at a system with m processors and n tasks to be solved. Each task is assigned to one of the m processors independently and uniformly, i.e., it is equally likely to be assigned to each of the processors.

a) What is the expected number of processors which have no task assigned to them?

b) What is the expected number of processors which have at least one task assigned to them?

c) What is the expected number of processors which have exactly one task assigned to them?

Task 3 – Expectation and Variance [6 pts]

Let X be a random variable with expected value μ and variance σ^2. Let $Y = (X - \mu)/\sigma$. Compute $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

Task 4 – Independence [10 pts]

We are given two independent random variables, M and K, both taking values in $\{0, 1, 2, 3, 4, 5\}$. Further, assume that you know that the marginal distribution of K is uniform, i.e., $\mathbb{P}(K = i) = 1/6$ for all $i \in \{0, 1, 2, \ldots, 5\}$. *You do not know the marginal distribution of M.* Finally, we define a new random variable C as

$$C = M + K \mod 6 \ .$$

a) For every $m, c \in \{0, 1, \ldots, 5\}$ such that $\mathbb{P}(M = m) > 0$, what is the probability

$$\mathbb{P}(C = c \mid M = m)\ ?$$

Hint: Write $\mathbb{P}(M = m, C = c)$ as a probability involving only K and M.

b) Show that M and C are (mutually) independent.