Homework 2

Due: Wednesday, Oct 16, by 11:59pm.
Refer to the instructions on Homework 1

Task 1 – Pigeons [8 pts]

Assume we pick 9 distinct integers from $[50] = \{1, \ldots, 50\}$, arbitrarily.

a) Show that there are two **distinct** subsets of the 9 integers whose elements have the same sum.

b) Show that there are two **disjoint** subsets of the 9 integers whose elements have the same sum.

Hint: Use **a)**.

Task 2 – Servers [8 pts]

A data center consists of 5 servers, $S_1, \ldots, S_5$. Each server fails independently with probability p during the day. Failed servers are restarted at the end of each day.

Give all of the following answers as a function of p – simplify the resulting expressions as far as possible.

a) What is the probability that S_1 or S_2 fails on a given day?

b) What is the probability that exactly two servers fail on a given day?

c) The data center remains operative as long as a server among $\{S_1, S_2, S_3\}$ is operative, *and* a server among $\{S_3, S_4, S_5\}$ is operative.

What is the probability that the data center fails on a given day?

d) A major re-design requires additionally that a server among $\{S_2, S_3, S_4\}$ must remain operative.

What is the probability that the data center fails on a given day (as a function of p)?

Task 3 – Hats [12 pts]

We consider a set with n people, who are each assigned, one by one, either a red or a blue hat uniformly at random. These choices are independent – the choice of the hat for one person is not affected by the choice of a hat for another person.

a) Describe the underlying probability space explicitly.

b) What is the probability that no person receives a red hat? (As a function of n.)

c) What is the probability that at most two people receive a red hat? (As a function of n.)

d) What is the probability that an even number of red hats are assigned? (As a function of n.)

Hint: In class, we have seen that $\sum_{\text{even } i} \binom{n}{i} = \cdots$?

e) Imagine every person can now see all of the remaining $n-1$ hats, except for their own. All n people will now try to guess their own hat (which they cannot see) simultaneously, *without* talking to each other.

Is there a strategy which allows them to be all correct with probability $1/2$?

Task 4 – Balls in an Urn [12 pts]

An urn contains 5 red, 7 green, and 8 blue balls. Three times in a row, we pick a random ball (uniformly) from the urn, and then discard it (i.e., we do not add it back to the urn). This is referred to as sampling *without* replacement.

a) What is the probability that all balls have different colors?

b) What is the probability that all balls have the same color?

c) If the first ball we get is red, what is the probability that the two remaining balls are blue?

d) If the first two balls have the same color, what is the probability that the third ball also has the same color?

e) Re-compute the above probabilities for the case of sampling *with* replacement, i.e., the ball is re-inserted into the urn after having been picked.