# Homework 1

Due: Wednesday, Oct 9, by 11:59pm.

## Instructions

**Solutions format.** Every step in your solution should be explained carefully. The logical reasoning behind your solution should be sound and evident from your write up.

For example, if you are asked to compute the number of ways to permute the set $\{1, 2, 3, 4\}$ that start with $1$ and $2$, it is not enough to provide the answer $12$. A complete approach would explain that (1) we can count separately the permutations starting with $1$ and those starting with $2$, and that (2) the two sets are disjoint, and hence the overall number is the sum of the numbers of permutations of each type. Then, (3) explain that there are $3!$ permutations of each type. Finally, (4) say that the overall number totals to $2 \cdot 3! = 12$.

A higher number of mathematical symbols in your solution will not make your solution more precise or "better" – what *is* important is that the logical flow is complete and can be followed by the graders. Relying exclusively on mathematical symbols can in fact often make the solution less readable. Avoid using expressions such as "it easy to see" and "clearly" – just explain these steps.

Also see the following short note: `https://www.math.hmc.edu/~su/math-writing.pdf` (by Francis E. Su at Harvey Mudd).

**Collaboration policy.** You are required to solve the homework alone. You are allowed to discuss the homework with other students in general terms. However, these discussions should not address the details of a solution. A general rule of thumb is that if your discussion is detailed enough for you to simply walk away and be able to write a solution, then you have gone too far.

You are allowed to use the textbook and other reading resources provided in class. You are not allowed to look up the solution to the task on the Internet, or anywhere else.

**Solutions submission.** You must submit your solution via Gradescope. You will be given instructions for signing up on the class's discussion forum. In particular:

- Submit a *single* PDF file containing the solution to all tasks in the homework. Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.
- Do not write your name on the individual pages – Gradescope will handle that. Also do not include a copy of the task's question.
- You can typset your solution (the homepage provides links to resources to help you doing so). You can also provide a handwritten solution, as long as it is on a single PDF file that satisfies the above submission format requirements. It is your responsibility to make sure handwritten solutions are readable – we will not grade unreadable write-ups.

## Task 1 – Counting Words [12 pts]

We want to count the number of strings of length $5$ from the English alphabet $\{A, B, \cdots, Z\}$ subject to a number of different constraints. Note that we consider the English alphabet here to consist of 6 *vowels* ($\{A, E, I, O, U, Y\}$) and 20 *consonants*.

How many strings are there which ...

**a)** ... are only made of vowels?

**b)** ... are only made of consonants?

**c)** ... have *exactly* one vowel?

**d)** ... have *exactly* two vowels?

**e)** ... have at most two vowels, which appear at the second and fourth position?

**f)** ... have at least one vowel?

**g)** ... have distinct vowels?

In all cases, explain your reasoning exactly – do not just give numbers or unjustified calculations.

## Task 2 – TAs [8 pts]

We need to assign the 7 CSE 312 TAs to the 5 available sections. How many ways are there to do so, if ...

**a)** ... the assignment is unrestricted (e.g., some sections may have $0$ TAs, some TAs may be assigned to all sections, ...)?

**b)** ... every section is assigned to exactly two TAs?

**c)** ... every TA is assigned to exactly two sections?

**d)** ... every TA is assigned to exactly two sections *and* every section is assigned to exactly two TAs?

## Task 3 – Binomial Coefficients and the Binomial Theorem [10 pts]

**a)** Use the Binomial Theorem to prove that for every even $n$,

$$1 + \sum_{i \in O} 2^i \cdot \binom{n}{i} = \sum_{i \in E} 2^i \cdot \binom{n}{i} ,$$

where $O$ and $E$ are the subsets of odd and even integers, respectively, between $0$ and $n$.

**b)** For integers $n \geqslant 0$, $k \geqslant 1$, give the number of sequences $(x_1, \ldots, x_k)$ of non-negative integers such that

$$\sum_{i=1}^{k} x_i \leqslant n .$$

**c)** For any integers $n \geqslant 0$ and $k \geqslant 1$, give a combinatorial proof of the identify

$$\sum_{i=0}^{n} \binom{i+k-1}{i} = \binom{n+k}{k} .$$

**Hint:** Find two different ways to count the elements of the same set which lead naturally to the left- and right-hand sides of the identity.

## Task 4 – Inclusion and Exclusion [10 pts]

There are three couples sitting at a round table with six seats. How many ways can the three couples be seated at the table if ...

**a)** ... there are no restrictions?

**b)** ... no one sits next to their partner (on either side)?

**Hint:** Use the inclusion-exclusion principle.

Note that here seating assignments are considered equivalent if one can be rotated to give the other one.