CSE 312: Foundations of Computing II
Quiz Section #7: Continuous Random Variables

Review: Main Theorems and Concepts

Continuous Random Variable: A r.v. which can take on an uncountably infinite number of values.

Probability Density Function (pdf or density): Let X be a continuous random variable. Then $f_X(x) : \mathbb{R} \to \mathbb{R}$ is a probability density function iff $\forall x\, f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Note that in general $f_X(x) \neq \mathbb{P}(X = x)$, since $\mathbb{P}(X = x) = 0$ for all x if X is continuous. However, the probability that X is close to x is proportional to $f_X(x)$: for small δ,

$\mathbb{P}\left(x - \frac{\delta}{2} < X < x + \frac{\delta}{2}\right) \approx \delta f_X(x)$.

Cumulative Distribution Function (cdf): For a continuous random variable X, $\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{dt}$ and therefore $F_X'(x) = f_X(x)$. Notice that $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$.

i.i.d. (independent and identically distributed): Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) if they are ____________ and have the same ______________________________.

Univariate: Discrete to Continuous:

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{dt}$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\,\mathrm{dx} = 1$
Expectation	$E[g(X)] = \sum_x g(x) p_X(x)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,\mathrm{dx}$

Zoo of Continuous Random Variables

Uniform: $X \sim Uni(a, b)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$. This represents each real number from $[a, b]$ to be equally likely.

Exponential: $X \sim Exp(\lambda)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$. $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. The exponential random variable is the continuous analog of the geometric random variable: it represents the waiting time to the next event, where $\lambda > 0$ is the average number of events per unit time. Note that the exponential measures how much time passes until the next event (any real number, continuous), whereas the Poisson measures how many events occur in a unit of time (nonnegative integer, discrete). The exponential random variable X is memoryless:

$$\text{for any } s, t \geq 0,\ \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

The geometric random variable also has this property.

Normal (Gaussian, "bell curve"): $X \sim N(\mu,\ \sigma^2)$ iff X has the following probability density function:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \mathbb{R}$$

$\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$. The "standard normal" random variable is typically denoted Z and has mean 0 and variance 1: if $X \sim N(\mu,\ \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. The CDF has no closed form, but we denote the CDF of the standard normal as $\Phi(z) = F_Z(z) = \mathbb{P}(Z \leq z)$. Note from symmetry of the probability density function about $z = 0$ that: $\Phi(-z) = 1 - \Phi(z)$.

Exercises

1. Alex decided he wanted to create a "new" type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We'll denote a random variable X having the "Uniform-2" distribution as $X \sim Unif2(a,b,c,d)$, where $a < b < c < d$. We want the density to be non-zero in $[a,b]$ and $[c,d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

   (a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\infty)$. (Hint: use a piecewise definition).

   (b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\infty)$. (Hint: use a piecewise definition).

2. Suppose $X \sim Unif(0,1)$ and $Y = e^X$. Find $f_Y(y)$.

3. A single-stranded (1-dimensional) spider web, with length W centimeters, where W>4, is stretched taut between two fence posts. The homeowner (a spider) sits precisely at the midpoint of this web. Suppose that a fly gets caught at a random point on the strand, with each point being equally likely.

   (a) The spider is lazy, and it is only willing to walk over and eat the fly if the fly lands within 2 centimeters of where the spider sits. What is the probability that the spider eats the fly?

   (b) Let X be the random variable that represents the spider's distance from the fly's landing point. Calculate the CDF, PDF, expectation, and variance of X.

4. Starting from the probability density function of $X \sim \text{Exp}(\lambda)$, prove that $\mathbb{E}[X] = 1/\lambda$. (Hint: use integration by parts.)

5. Starting from the probability density function of $X \sim \text{Exp}(\lambda)$, prove that $\mathbb{P}(X \geq t) = e^{-\lambda t}$, for $t \geq 0$. As a corollary, show that the cumulative distribution function for X is $F_X(t) = 1 - e^{-\lambda t}$.

6. Prove the memorylessness property for the exponential distribution $\text{Exp}(\lambda)$: If s and t are nonnegative, then $\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$.

7. Prove the memorylessness property for the geometric distribution $\text{geo}(p)$.

8. Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f(x) = \frac{c}{1+x^2}$ is valid. Then find $\mathbb{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

9. Let $X \sim \text{Exp}(\lambda)$. For $t < \lambda$, find $M_X(t) = \mathbb{E}[e^{tX}]$. M is called the *moment generating function* of X. Find $M'_X(0)$ and $M''_X(0)$. Do you notice any relationship between these two values and $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ (which are sometimes called the first and second *moments* of X)?

10. You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest square B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For X, find the CDF, PDF, E[X], and Var(X).

11. You throw a dart at a one-dimensional target whose bullseye is the origin. Let X be the distance of your dart's hit from the origin. You are very good at darts and your aim is such that $X \sim \text{Exp}(0.5)$.

    (a) Determine the value m such that $P(X < m) = P(X > m)$. Give your answer to 3 significant digits.

    (b) The target has radius 8 rather than being the infinite line. What is the probability that you miss the target completely? Give your answer to 3 significant digits.