CSE 312: Foundations of Computing II
Quiz Section #5: Midterm review

1. Let A and B be events in the same sample space that each have nonzero probability. For each of the following statements, state whether it is always true, always false, or it depends on information not given.

    (a) If A and B are mutually exclusive, then they are independent.

    (b) If A and B are independent, then they are mutually exclusive.

    (c) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are mutually exclusive.

    (d) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are independent.

2. Given any set of 18 integers, show that one may always choose two of them so that their difference is divisible by 17.

3. (This problem is similar to a problem from HW1, except that "=" has been replaced by "≤".)

    Consider the following inequality: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \le 70$. A solution to this inequality over the nonnegative integers is a choice of a nonnegative integer for each of the 6 variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the inequality. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the inequality?

4. You roll three fair dice, each with a different numbers of faces: die 1 has six faces (numbered 1 ... 6), die 2 has eight faces (numbered 1 ... 8), and die 3 has twelve faces (numbered 1 ... 12). Let the random variable X be the sum of the three values rolled. What is $\mathbb{E}[X]$?

5. How many integers in $\{1, 2, \ldots, 360\}$ are divisible by one or more of the numbers 2, 3, and 5?

6. Recall that a Schnapsen deck has 4 suits with 5 cards in each suit. Suppose a deck of Schnapsen cards is shuffled well and then dealt into 5 piles of 4 cards each. Let E_i refer to the event that pile i has exactly one spade. Compute the probability $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5)$.

7. You are trying to diagnose the probability that a patient with a positive blood sugar test result has diabetes, even though she is in a low risk group. The probability of a woman in this group having diabetes is 0.8%. 90% of women with diabetes will test positive in the blood sugar test. 7% of women without diabetes will test positive in the blood sugar test. Your patient tests positive in the blood sugar test. What is the probability that she has diabetes?

8. A very long multiple choice exam has 4 choices for each question. Charlie has studied enough so that he knows the correct answer for 1/2 of the questions; for an additional 1/4 of the questions he can eliminate one choice and chooses randomly and uniformly among the other three, and for the remaining 1/4 of the questions he chooses randomly and uniformly among all four answers.

As the teacher, you want to determine how many answers the student actually knows. For a randomly chosen question, if Charlie answers it correctly, what is the probability he knew the answer?

9. The space shuttle has 6 O-rings: these were involved in the Challenger disaster. When the space shuttle is launched, each O-ring has a probability of failure of 0.0137, independent of whether other O-rings fail.

    (a) What is the probability that, during 23 launches, no O-ring will fail, but that at least one O-ring will fail during the 24th launch?

    (b) What is the probability that no O-ring fails during 24 launches?

10. Suppose you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur? Assume that there are 365 possible birthdays and each one is equally probable for a randomly chosen person.

11. Two fair 6-sided dice are thrown n times in succession. Compute the probability that double 6 (i.e., 6 on each die) appears at least once in the n throws.

    How large need n be to make this probability at least 1/2?

12. You are working on a difficult passage from a new piece you are learning on the piano. You wish to play it correctly 4 times before stopping for the day. If your probability of playing it correctly on each attempt is 2/3, and the attempts are independent (unfortunately!), what is the probability that you have to play it at least 8 times?

13. The probability that a customer pays with cash is 40%, independent of other customers. Find the probability that the 12th customer to arrive at the cashier is the 8th one that pays with cash.

14. Let X be the outcome of rolling a fair 6-sided die once. Let Y be the sum of the outcomes of rolling the same die n times independently.

    (a) Compute $\mathbb{E}[X]$.

    (b) Compute $\text{Var}(X)$ and the standard deviation σ of X.

    (c) Compute $\mathbb{E}[Y]$.

    (d) Compute $\text{Var}(Y)$.

15. For $n > 1$, let $a_1, ..., a_n \in [0, 1]$. Show that there exist numbers $x_1, ..., x_n \in \{-1, 0, 1\}$ not all zero such that $\left| \sum_{i=1}^n a_i x_i \right| \le \dfrac{n}{2^n - 2}$.