Ex: Independent samples $x_1, x_2, \ldots, x_n \sim N(\mu, \sigma^2)$, where $\theta_1 = \mu$ and $\theta_2 = \sigma^2$ are both unknown.

$$L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$\ln L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} \left(-\frac{1}{2}\ln(2\pi\theta_2) - \frac{(x_i-\theta_1)^2}{2\theta_2} \right)$$

$$\frac{\partial}{\partial\theta_1} \ln L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} + \frac{2(x_i-\theta_1)}{2\theta_2} = \boxed{\sum_{i=1}^{n} \frac{x_i-\theta_1}{\theta_2}} = 0$$

$$\sum_{i=1}^{n} (x_i - \hat{\theta}_1) = 0$$

$$\sum_{i=1}^{n} x_i - n\hat{\theta}_1 = 0$$

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i, \text{ the sample mean}$$

Is it a max?

$$\frac{\partial^2}{\partial\theta_1^2} \ln L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} -\frac{1}{\theta_2} = \frac{-n}{\theta_2} < 0$$

so $\ln L$ is concave downward everywhere.

$$\frac{\partial}{\partial\theta_2} \ln L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} \left(-\frac{2\pi}{2 \cdot 2\pi\theta_2} + \frac{(x_i-\theta_1)^2}{2\theta_2^2} \right)$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{2\theta_2} + \frac{(x_i-\theta_1)^2}{2\theta_2^2} \right) = 0$$

$$\sum_{i=1}^{n} \left(-\hat{\theta}_2 + (x_i - \hat{\theta}_1)^2 \right) = 0$$

$$-n\hat{\theta}_2 + \sum_{i=1}^{n} (x_i - \hat{\theta}_1)^2 = 0$$

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\theta}_1)^2$$

2^{nd} derivative test shows that it is negative at $\theta_2 = \hat{\theta}_2$.

$\hat{\theta}_2$ is a "sample variance".

$$Var(X) = E[(X-\mu)^2] \text{ where } \mu = E[X].$$

MLE of population variance $\hat{\sigma}^2$ is a sample variance of $x_1, x_2, \cdots, x_n$.

Bias

Defn: An estimator $\hat{\theta}$ of θ is unbiased iff $E[\hat{\theta}] = \theta$.

Ex: For $\hat{\theta}_1$ of $N(\theta_1, \theta_2)$, is $\hat{\theta}_1$ unbiased?

$$E[\hat{\theta}_1] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[x_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}\cdot n\mu = \mu = \theta_1$$

So $\hat{\theta}_1$ is an unbiased estimator.

Is $\hat{\theta}_2$ an unbiased estimator of θ_2? No

$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\theta}_1)^2$ ~~has~~ maximizes likelihood

$\hat{\theta}_2' = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\theta}_1)^2$ is an unbiased estimator of θ_2.

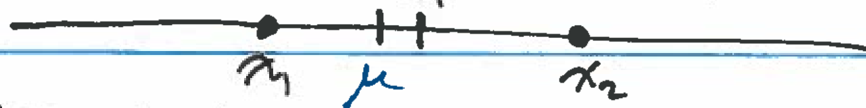
$\hat{\theta}_2$ and $\hat{\theta}_2'$ differ by a factor of $\frac{n-1}{n}$.

As n gets large, they get closer.

why might $\hat{\theta}_2$ be biased?

$n = 2$

$\hat{\theta}_1 = \frac{1}{2}(x_1 + x_2)$



$$\hat{\theta}_2 = \frac{1}{2}\left((x_1 - \hat{\theta}_1)^2 + (x_2 - \hat{\theta}_1)^2\right)$$

$$< \frac{1}{2}\left((x_1 - \mu)^2 + (x_2 - \mu)^2\right)$$

so $\hat{\theta}_2$ underestimates σ^2