Chebyshev's Inequality

<u>Theorem</u>: If Y is a r.v. with $E[Y]=\mu$, then for any $\alpha>0$, $P(|Y-\mu|\geq\alpha)\leq\dfrac{Var(Y)}{\alpha^2}$.

Equivalently, if $Var(Y)=\sigma^2$, then for any $k>0$,

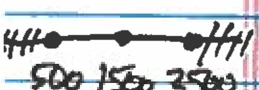
$$P(|Y-\mu|\geq k\sigma)\leq\frac{1}{k^2}.$$

<u>Proof</u>: Let $X=(Y-\mu)^2$. X is nonnegative, so

$$P(|Y-\mu|\geq\alpha)=P(X\geq\alpha^2)\leq\frac{E[X]}{\alpha^2}\quad\text{(Markov)}$$

$$=\frac{Var(Y)}{\alpha^2}$$

<u>Ex</u>: Suppose Y is your daily business cost, and $E[Y]=1500$ and $\sigma_Y=200$. Let $\mu=E[Y]$.



$$P(Y\geq2500)=P(Y-1500\geq1000)\leq P(|Y-1500|\geq1000)$$

$$\leq\frac{Var(Y)}{1000^2}=\frac{200^2}{1000^2}=\left(\frac{1}{5}\right)^2=\frac{1}{25}$$

<u>Cantelli's Inequality</u> (One-sided Chebyshev):
If $\alpha>0$, and $\mu=E[Y]$, then

$$P(Y-\mu\geq\alpha)\leq\frac{Var(Y)}{Var(Y)+\alpha^2}.$$

In the example above,

$$P(Y\geq2500)\leq\frac{200^2}{200^2+1000^2}=\frac{1}{1+25}=\frac{1}{26}$$

Chernoff Bound:

Theorem: Suppose $X \sim Bin(n,p)$ and $\mu = E[X] = np$. For any $0<\delta<1$,

$$P(X \geq (1+\delta)\mu) \leq e^{-\frac{1}{3}\delta^2\mu} \quad \text{and}$$

$$P(X \leq (1-\delta)\mu) \leq e^{-\frac{1}{2}\delta^2\mu}$$

Law of Large Numbers:

Consider i.i.d. r.v.'s $X_1, X_2, X_3, \ldots$, where $E[X_i] = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$.

Define sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$$E[\overline{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\cdot n\mu = \mu$$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

Theorem (Weak Law of Large Numbers):

For any $\varepsilon > 0$, as $n \to \infty$,

$$P(|\overline{X}_n - \mu| > \varepsilon) \to 0.$$

Proof: By Chebyshev's Inequality,

$$P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$