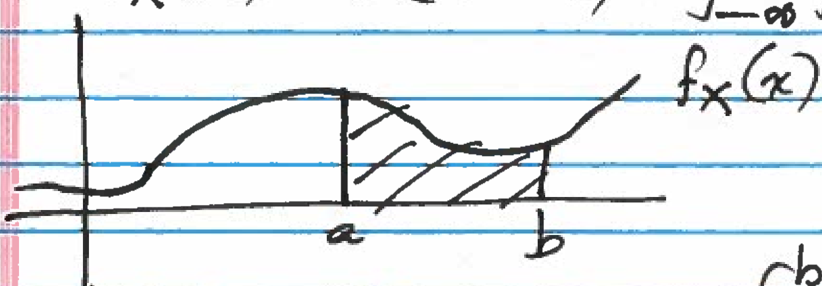
Continuous random variables

set of values of the r.v. is uncountable.
For us, either $\mathbb{R}$ or an interval of the $\mathbb{R}$.

Ex: time until next request arrives at a server.
height of a random adult

Defn: $f_X : \mathbb{R} \to \mathbb{R}$ is a probability density function (or PDF or density) for continuous r.v. X iff $\forall x \; f_X(x) \geq 0$, and
$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$.

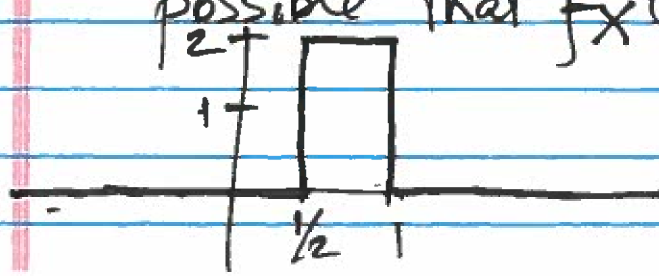
Defn: The cumulative distribution function (CDF) $F_X : \mathbb{R} \to \mathbb{R}$ associated with r.v. X is
$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\,dx$



Note: $P(a < X \leq b) = \int_a^b f_X(x)\,dx$ for $a \leq b$.
$= F_X(b) - F_X(a)$

Note: $f_X(x) = \frac{d}{dx} F_X(x)$

Note: Densities are not probabilities! It is possible that $f_X(a) > 1$.

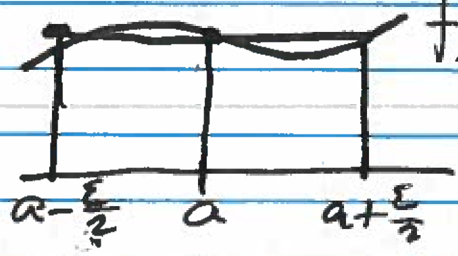


Note: $P(X=a) = F_X(a) - F_X(a) = 0$.

But $P(a - \frac{\varepsilon}{2} < X \le a + \frac{\varepsilon}{2})$

$$= F_X(a+\tfrac{\varepsilon}{2}) - F_X(a-\tfrac{\varepsilon}{2}) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f_X(x)\,dx$$

$$\approx f_X(a)\cdot\varepsilon$$



That is, the probability that X is "near" a is proportional to $f_X(a)$.

For continuous r.v.'s, usually substitute $\int$ for Σ, and substitute f_X for p_X. For instance,

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx.$$

$$E[aX+b] = a\,E[X] + b$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

$$Var(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2,$$

where $\mu = E[X]$.

X and Y are <u>independent</u> iff

$$\forall A\,\forall B\; P(X\in A \cap Y\in B) = P(X\in A)P(Y\in B).$$

(discrete: $\forall x\,\forall y\; P(X=x \cap Y=y) = P(X=x)P(Y=y)$)

Important classes of continuous r.v.'s

Uniform:

For $\alpha < \beta$, $X \sim Uni(\alpha, \beta)$ iff

$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } x \in [\alpha, \beta] \\ 0, & \text{otherwise} \end{cases}.$$

See previous diagram for $Uni(\frac{1}{2}, 1)$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_\alpha^\beta x \cdot \frac{1}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha}\int_\alpha^\beta x\,dx = \frac{1}{\beta-\alpha} \cdot \frac{1}{2}x^2 \Big|_\alpha^\beta$$

$$= \frac{1}{2} \cdot \frac{\beta^2-\alpha^2}{\beta-\alpha} = \frac{1}{2}(\alpha+\beta).$$