Theorem: If X and Y are independent r.v.'s, then $E[XY] = E[X]E[Y]$.

Proof: $E[XY] = \sum_x \sum_y xy\, P(X=x \cap Y=y)$

$= \sum_x \sum_y xy\, P(X=x)P(Y=y)$ (ind.)

$= \sum_x x P(X=x)\left(\sum_y y\, P(Y=y)\right)$

$= \left(\sum_x x P(X=x)\right)\left(\sum_y y\, P(Y=y)\right)$

$= E[X]E[Y]$.

Theorem: If X and Y are independent r.v.'s, then $Var(X+Y) = Var(X) + Var(Y)$.
(Additivity of variance).

Proof: $Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$

$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$

$= E[X^2] + 2E[XY] + E[Y^2]$
$\quad - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2$

$= (E[X^2] - (E[X])^2) + (E[Y^2] - (E[Y])^2)$
$\quad + 2E[X]E[Y] - 2E[X]E[Y]$ (ind)

$= Var(X) + Var(Y)$

Some important discrete r.v.s

Defn: A Bernoulli random variable $X \sim Ber(p)$ is a random indicator variable with $P(X=1)=p$ and $P(X=0)=1-p$.

Ex: One flip of a coin.

$$E[X] = 1 \cdot p + 0(1-p) = p$$
$$E[X^2] = 1^2 \cdot p + 0^2(1-p) = p$$
$$Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p).$$

Defn: A binomial random variable $X \sim Bin(n,p)$ is the sum of n independent $Ber(p)$ r.v.'s.

Ex: # of heads in n independent flips of a coin with prob. p of heads.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$E[X] = np$$

Let $X_1, X_2, \ldots, X_n \sim Ber(p)$ be independent.

Let $X = \sum_{i=1}^{n} X_i$.

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) \quad \text{(additivity of ind. r.v.'s)}$$
$$= \sum_{i=1}^{n} p(1-p) = np(1-p).$$

Defn: X is a uniform random variable, $X \sim Unif(a,b)$, if X is equally likely to be any integer in $[a,b]$.

Ex: One roll of a fair die is $Unif(1,6)$.

$$P(X=i) = \begin{cases} \frac{1}{b-a+1}, & \text{if } i \in \{a, a+1, \dots, b\} \\ 0, & \text{otherwise.} \end{cases}$$

$$E[X] = \tfrac{1}{2}(a+b)$$

$$Var(X) = \tfrac{1}{12}(b-a)(b-a+2)$$

Ex: for $X \sim Unif(1,6)$,

$$E[X] = 7/2$$

$$Var(X) = 35/12.$$