Variance fair coin

Consider two games between players A and B:

1. A's gain per flip is $X = \begin{cases} +1 & \text{if heads} \\ -1 & \text{if tails} \end{cases}$

2. A's gain per flip is $Y = \begin{cases} +1000 & \text{if heads} \\ -1000 & \text{if tails} \end{cases}$

$E[X] = E[Y] = 0$

Game 2 seems scarier, but how do we measure that? Variability.

Defn: Let X be a r.v. with $E[X] = \mu$. The variance of X is $Var(X) = E[(X-\mu)^2]$, often denoted σ^2.

Defn: The standard deviation of X is $\sigma = \sqrt{Var(X)}$.

Back to our two games:

$Var(X) = 1^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$

$Var(Y) = 1000^2 \cdot \frac{1}{2} + (-1000)^2 \cdot \frac{1}{2} = 1{,}000{,}000$

Examples from slides 33 & 35 of slide pack 6.

Theorem: $Var(X) = E[X^2] - (E[X])^2$.

Proof: Let $\mu = E[X]$.

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - 2\mu E[X] + \mu^2 \quad \text{(linearity of E)} \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}
$$

Theorem: $Var(aX+b) = a^2 Var(X)$. Let $\mu = E[X]$

Proof: $Var(aX+b) = E[((aX+b)-(a\mu+b))^2]$
$= E[(a(X-\mu))^2]$
$= E[a^2(X-\mu)^2]$
$= a^2 E[(X-\mu)^2]$
$= a^2 Var(X)$.

In general, $Var(X+Y) \neq Var(X) + Var(Y)$.
($Var(X+X) = Var(2X) = 4Var(X) \neq Var(X) + Var(X)$

Defn: R.V.'s X and Y are independent iff
$\forall x \forall y \; P(X=x \cap Y=y) = P(X=x)P(Y=y)$.

Ex: Flip a fair coin $2n$ time independently.
X = # of heads in first n flips.
Y = # of heads in ~~second~~ last n flips.
Z = # of heads in all $2n$ flips.
X and Y are independent.
X and Z are dependent:
$P(X=0) > 0$, $P(Z=n+1) > 0$
but $P(X=0 \cap Z=n+1) = 0$