Theorem: Let X and Y be two random variables, possibly dependent. Then $E[X+Y] = E[X]+E[Y]$.

Proof: Let $X(s)$ and $Y(s)$ be the values for outcome $s \in \Omega$.

$$E[X+Y] = \sum_{s\in\Omega} (X(s)+Y(s))P(s)$$
$$= \sum_{s\in\Omega} (X(s)P(s)+Y(s)P(s))$$
$$= \sum_{s\in\Omega} X(s)P(s) + \sum_{s\in\Omega} Y(s)P(s)$$
$$= E[X] + E[Y]$$

Let X be the number of heads in n independant flips of a coin, where each flip is heads with prob. p. Compute $E[X]$. (X is a "binomial" r.v.)

Let $X_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ flip is heads} \\ 0, & \text{otherwise} \end{cases}$ for $1 \le i \le n$.

(X_i is called an indicator r.v.: values $\in \{0,1\}$.)

$$E[X_i] = 1 \cdot P(X_i=1) + 0 \cdot P(X_i=0).$$
$$= P(i^{\text{th}} \text{ flip is heads}) = p.$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] \quad \text{(linearity)}$$
$$= \sum_{i=1}^{n} p = pn.$$

Ex: Quiz Section 4 worksheet, exercise 9

Shuffle 4 aces, deal 2 face-down.
Let X be the number of spades in the 2 piles.

Let $X_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ card is } \spadesuit A \\ 0, & \text{otherwise} \end{cases}$, $i=1,2$

$$E[X_i] = 1\cdot P(X_i=1) + 0\cdot P(X_i=0)$$
$$= P(i^{\text{th}} \text{ card is } \spadesuit A)$$
$$= 1/4$$

$$X = X_1 + X_2$$
$$E[X] = E[X_1+X_2] = E[X_1] + E[X_2] \quad \text{(linearity)}$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

You could compute $E[X_2]$ the hard way:
$$P(X_2=1) = P(X_2=1 \mid X_1=1)P(X_1=1) + P(X_2=1 \mid X_1=0)P(X_1=0)$$
$$= 0\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$$

But $P(X_2=1) = \frac{1}{4}$ obviously, as though the 2nd was dealt first.

Linearity is special. In general,
$$E[XY] \neq E[X]E[Y]$$
$$E[X^2] \neq (E[X])^2$$
$$E[\sqrt{X}] \neq \sqrt{E[X]}$$
etc.