Defn: A *random variable* is a numeric function of the outcome.

Ex: # of heads when 20 coins are flipped.

Ex: Sum of 2 die rolls.

Ex: # of coin tosses until the first head.

Defn: If a random variable has a countable number of possible outcomes, it is called *discrete*.

(Countable means either finite or in one-to-one correspondence with the nonnegative integers.)

Defn: If X is a discrete r.v. with values in countable set T, the *probability mass function (pmf)* of X is

$$p_X(a) = \begin{cases} P(X=a), & \text{if } a \in T \\ 0 & \text{, otherwise} \end{cases}$$

Note: $\sum_{a \in T} p_X(a) = 1.$

Defn: For a discrete r.v. X with pmf p_X, the *expectation* (or *expected value* or *mean*) of X is $E[X] = \sum_x x\, p_X(x)$.

In the case of equally likely values, this is just the average value. In the general case, it is a weighted average of the values, weighted by their probabilities.

n independent flips of coin with prob p of heads:
For the graphs in slide 13 of 06rvs.pdf,

$$E[X] = \sum_{k=0}^{n} k\, p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

This can be simplified, but how? Next Monday.

New example:
Let Y be the number of independent flips of a coin up to and including the first head, where prob of head is p.
Y is called a geometric random variable.

$$p_Y(i) = (1-p)^{i-1} p, \quad \text{for } i \in \{1, 2, 3, \dots\}$$

$$E[Y] = \sum_{i=1}^{\infty} i\, p_Y(i) = \sum_{i=1}^{\infty} i (1-p)^{i-1} p$$

$$= p \sum_{i=1}^{\infty} i (1-p)^{i-1}$$

Calculus: $\displaystyle\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, for $|x|<1$

$$= (1-x)^{-1}$$

Differentiate: $\displaystyle\sum_{i=0}^{\infty} i x^{i-1} = +(1-x)^{-2} = \frac{1}{(1-x)^2}$

$$\sum_{i=1}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2}, \text{ for } |x|<1.$$

$$E[Y] = p \sum_{i=1}^{\infty} i (1-p)^{i-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

$p = \frac{1}{2} \Rightarrow E[Y] = 2$

$p = \frac{1}{10} \Rightarrow E[Y] = 10$