Independence

Defn: Two events E and F are independent iff $P(E\cap F) = P(E)P(F)$. Otherwise they are dependent.

Ex: Roll 2 fair dice, yielding values D_1 and D_2. Let $E =$ "$D_1 = 1$", $F =$ "$D_1 + D_2 = 7$", $G =$ "$D_1 + D_2 = 5$".

$P(E) = \frac{1}{6}$

$P(F) = \frac{6}{36} = \frac{1}{6}$

$P(E\cap F) = P(D_1 = 1 \cap D_2 = 6) = \frac{1}{36} = P(E)P(F)$,

so E and F are independent.

$P(E\cap G)$

$P(G) = \frac{4}{36} = \frac{1}{9}$

$P(E\cap G) = P(D_1 = 1, D_2 = 4) = \frac{1}{36} \neq \frac{1}{6}\cdot\frac{1}{9} = P(E)P(G)$,

so E and G are dependent.

Defn: $E_1, E_2, \cdots, E_n$ are independent iff for every subset S of $\{1, 2, \cdots, n\}$

$$P\left(\bigcap_{i\in S} E_i\right) = \prod_{i\in S} P(E_i).$$

Ex: Let X, Y each be ± 1 with probability $\frac{1}{2}$ independently. Let $E =$ "$X = 1$", $F =$ "$Y = 1$", $G =$ "$XY = 1$".

These are pairwise independent, but

$P(E\cap F\cap G) = P(X=1 \cap Y=1) = \frac{1}{4} \neq \frac{1}{8}$

$= P(E)P(F)P(G)$

so E, F, G are not independent.

<u>Theorem</u>: If $P(F) > 0$, then E and F are independent iff $P(E|F) = P(E)$.

<u>Proof</u>:

$\Rightarrow$: E and F independent $\Rightarrow P(E\cap F) = P(E)P(F)$

$$\Rightarrow P(E\cap F) = P(E|F)P(F) = P(E)P(F)$$

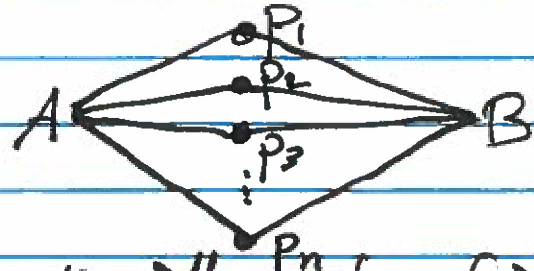
$$\Rightarrow P(E|F) = P(E)$$

$\Leftarrow$: $P(E|F) = P(E) \Rightarrow \dfrac{P(E\cap F)}{P(F)} = P(E)$

$$\Rightarrow P(E\cap F) = P(E)P(F) \Rightarrow E, F \text{ indep.}$$

<u>Network failure</u>:

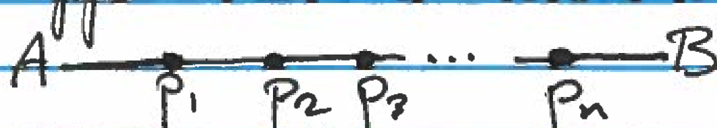
1. Suppose there are n routers in parallel:



Suppose the i^{th} router fails with probability p_i, independently of the others.
Let E = "A can communicate with B".

$$P(E) = 1 - P(\text{all } n \text{ routers fail})$$
$$= 1 - p_1 p_2 \cdots p_n.$$

2. Suppose there are n routers in series:



$$P(E) = P(\text{none of the } n \text{ routers fail})$$
$$= (1-p_1)(1-p_2)\cdots(1-p_n).$$

Ex. Suppose a coin comes up heads with prob. p and tails with prob. $1-p$. Suppose it is flipped n times independently.

$P(n \text{ heads}) = p^n$

$P(\text{first } k \text{ flips are heads and remaining tails})$
$$= p^k (1-p)^{n-k}$$

$P(\text{exactly } k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$

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$P(0 \text{ heads or } 1 \text{ head or} \cdots \text{or } n \text{ heads})$
$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+(1-p))^n \quad \text{(Bin. Thm.)}$$
$$= 1^n = 1$$