Let E_i = A wins starting with # i.
What is $P(E_i)$? Let $p_i = P(E_i)$.
Condition on outcome of the first flip, and use the LTP.

$$p_i = P(E_i) = P(E_i \mid H)P(H) + P(E_i \mid T)P(T)$$
$$= P(E_{i+1}) \cdot \tfrac{1}{2} + P(E_{i-1}) \cdot \tfrac{1}{2}$$
$$= \tfrac{1}{2}(p_{i+1} + p_{i-1})$$

$$2p_i = p_{i+1} + p_{i-1}$$
$$p_i - p_{i-1} = p_{i+1} - p_i$$
$$p_2 - p_1 = p_1 - p_0 = p_1 - 0 = p_1$$
$$p_2 = 2p_1$$
$$p_3 = 3p_1$$
$$p_4 = 4p_1$$
$$p_i = i p_1 \text{ for } 0 \le i \le N.$$
$$1 = p_N = N p_1$$
$$p_1 = 1/N$$
$$p_i = i/N$$

Bayes' Theorem (Rev. Thomas Bayes, c.1701–1761):

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)}$$

Proof: $P(F|E) = \frac{P(E\cap F)}{P(E)} = \frac{P(E|F)P(F)}{P(E)}$

Corollary: $P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})}$

why it's useful:
$P(\text{disease} \mid \text{test result}) \sim P(\text{test result} \mid \text{disease})$

Ex: 60% of email is spam.
90% of spam has a forged header
20% of nonspam has a forged header.
~~Wha~~ Let F = forged header, J = spam
What is $P(J|F)$?

$$P(J|F) = \frac{P(F|J)\,P(J)}{P(F|J)P(J) + P(F|\bar{J})P(\bar{J})}$$

$$= \frac{0.9\times 0.6}{0.9\times 0.6 + 0.2\times(1-0.6)} \approx 0.871$$

"prior" $P(J) = 0.6$
"posterior" $P(J|F) \approx 0.871$

Ex: Paternity testing

Child has (A, a) gene pair : event B_{Aa}

Mother has (A, A)

Two possible fathers, $F_1 = (a,a)$, $F_2 = (A,a)$.

$P(F_1) = p$, $P(F_2) = 1-p$.

$$P(F_1 | B_{Aa}) = \frac{P(B_{Aa} | F_1) P(F_1)}{P(B_{Aa}|F_1) P(F_1) + P(B_{Aa}|F_2) P(F_2)}$$

$$= \frac{1 \cdot p}{1 \cdot p + \frac{1}{2}(1-p)} = \frac{2p}{2p + 1 - p} = \frac{2p}{1+p}$$

$$\frac{2p}{1+p} \geq \frac{2p}{1+1} = p \text{ . Eg., } p = \frac{1}{2}, \ \frac{2p}{1+p} = \frac{2}{3}$$