Ex: Roll a fair die. What is $P(5 \mid odd)$

$E = \{5\}$, $F = \{1,3,5\}$

$$P(E|F) = \frac{|E\cap F|}{|F|} = \frac{|E|}{|F|} = \frac{1}{3}$$

$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Ex: Assume ♠J is the face-up trump. Let Ω be the set of ways of dealing 2 Schnapsen hands, let Y = you have no trumps and let θ = opponent has no trumps.

$$P(\theta|Y) = \frac{|\theta\cap Y|}{|Y|} = \frac{\binom{15}{5}\binom{10}{5}}{\binom{15}{5}\binom{14}{5}}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{14\cdot 13\cdot 12\cdot 11\cdot 10} \approx 0.126$$

Compare to $P(Y) \approx 0.258$

$$P(\bar{\theta}|Y) = 1 - P(\theta|Y) \approx 1 - 0.126 = 0.874$$

Outcomes not all equally likely.

Ex: Let $\Omega = \{0,1,2\}$, the number of heads when a fair coin is flipped twice.

$P(0) = P(2) \triangleq 1/4$, $P(1) = 1/2$

General defn of conditional probability:

If $P(F) \neq 0$, then $P(E|F) = \dfrac{P(E\cap F)}{P(F)}$.

Ex: Let E be "2 heads" and F be "≥ 1 head".

$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Chain rule: $P(E \cap F) = P(F)P(E|F)$

Generalized:

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2) \cdots P(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1})$$

Law of Total Probability: If E and F are events, then $P(E) = P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$.

Proof:
$$P(E) = P((E \cap F) \cup (E \cap \bar{F}))$$
$$= P(E \cap F) + P(E \cap \bar{F})$$
$$= P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$$
$$\left(= P(E|F)P(F) + P(E|\bar{F})(1 - P(F))\right)$$

Ex: Sally will take either physics or chemistry. She will get an A in physics with probability $3/4$ and an A in chemistry with probability $3/5$. She flips a fair coin to decide which course to take. Let A be the event she gets an A.

$$P(A) = P(A|\text{Phys})P(\text{Phys}) + P(A|\text{Chem})P(\text{Chem})$$
$$= \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{27}{40}.$$

Gambler's Ruin: A has \$$i$ and B has \$$(N-i)$. Flip a fair coin: $H \Rightarrow$ A wins \$1 from B, $T \Rightarrow$ B wins \$1 from A. The first person to get all \$$N$ wins.

Random walk on the line

Let E_i = A wins starting with # i.
What is $P(E_i)$? Let $p_i = P(E_i)$.
Condition on outcome of the first flip, and use the LTP.

$$p_i = P(E_i) = P(E_i \mid H)P(H) + P(E_i \mid T)P(T)$$
$$= P(E_{i+1})\cdot \tfrac{1}{2} + P(E_{i-1})\cdot\tfrac{1}{2}$$
$$= \tfrac{1}{2}(p_{i+1} + p_{i-1})$$