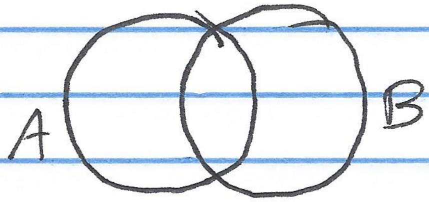
<u>Corollary</u>: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

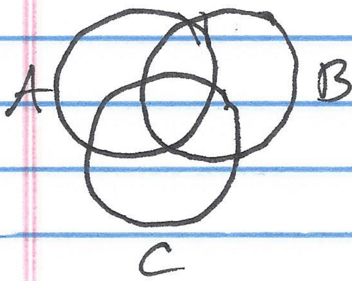
<u>Proof</u>: $2^n = (1+1)^n \underset{\uparrow}{=} \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

Binomial Theorem

<u>Inclusion</u>-<u>Exclusion</u>:



$|A \cup B| = |A| + |B| - |A \cap B|$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

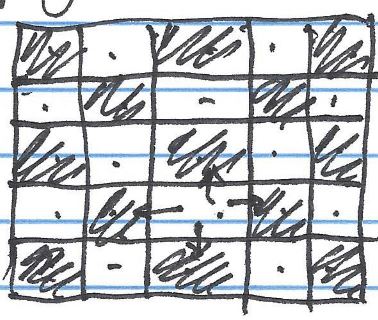
In general, + singles − pairs + trips − quads ....
Back to example of # of ways to deal at least 1 trump in Schnapsen.

$$\binom{4}{1}\binom{18}{4} - \binom{4}{2}\binom{17}{3} + \binom{4}{3}\binom{16}{2} - \binom{4}{4}\binom{15}{1}$$

$= 12{,}240 \quad - \quad 4080 \quad + \quad 480 \quad - \quad 15$

$= 8625$

Pigeonhole Principle: If you have n pigeons and k pigeonholes, where $n > k$, and every pigeon is in a pigeonhole, then some pigeonhole has at least 2 pigeons.



You place 1 flea on each square of a 5×5 chessboard. When you ring a bell, all 25 jump to an adjacent square. Some square has 2 fleas now.

pigeons = fleas on black squares, $n = 13$
pigeonholes = white squares, $k = 12$

Intro to Probability

A sample space is a set Ω of possible outcomes of a probabilistic experiment.

Ex: Coin flip: $\Omega = \{H, T\}$.
2 coin flips: $\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$
die roll: $\Omega = \{1,2,3,4,5,6\}$
Initial Schnapsen hands from a well-shuffled deck, with ♣J as the trump showing: $|\Omega| = \binom{19}{5}$.

An event E is a subset of Ω.

Ex: ≥ 1 head when 2 coins flipped: $E = \{(H,H), (H,T), (T,H)\}$
odd roll of die: $E = \{1, 3, 5\}$.
Schnapsen hands with 0 trumps: $|E| = \binom{15}{5}$.

<u>Defn</u>: E and F are <u>mutually exclusive</u> iff $E \cap F = \emptyset$.

<u>Axioms of Probability</u>:
There is a function P that assigns a real number $P(E)$ to every event E satisfying:
(1) $P(E) \geq 0$,
(2) $P(\Omega) = 1$, and
(3) If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

<u>Corollaries</u>:
(a) $P(\overline{E}) = 1 - P(E)$, because
$$1 \underset{\text{Ax 2}}{=} P(\Omega) = P(E \cup \overline{E}) \underset{\text{Ax 3}}{=} P(E) + P(\overline{E})$$

(b) If $E \subseteq F$, then $P(E) \leq P(F)$, because
$$P(F) = P(E \cup (F-E)) \underset{\text{Ax 3}}{=} P(E) + P(F-E)$$
$$\underset{\text{Ax 1}}{\geq} P(E) + 0 = P(E)$$
(c) $P(E) \leq 1$, because $E \subseteq \Omega$ and $P(\Omega) = 1$.
(d) $P(E \cup F) \stackrel{!}{=} P(E) + P(F) - P(E \cap F)$.

<u>Equally likely outcomes</u>: Ω is finite, and for each $a \in \Omega$, $P(a) = 1/|\Omega|$.

<u>Ex:</u> flip a fair coin: $P(H) = P(T) = 1/2$.
flip 2 fair coins: $P(HH) = 1/4$, etc.
roll a fair die: $P(a) = 1/6$

In the case of equally likely outcomes,

$$P(E) = \frac{|E|}{|\Omega|}. \quad \text{(Axiom 3)}.$$