Combinations: order doesn't matter.

Ex. Your elf-lord avatar can pick up any 3 of the following weapons:
sword, axe, knife, ring, laptop
How many combinations of exactly 3?

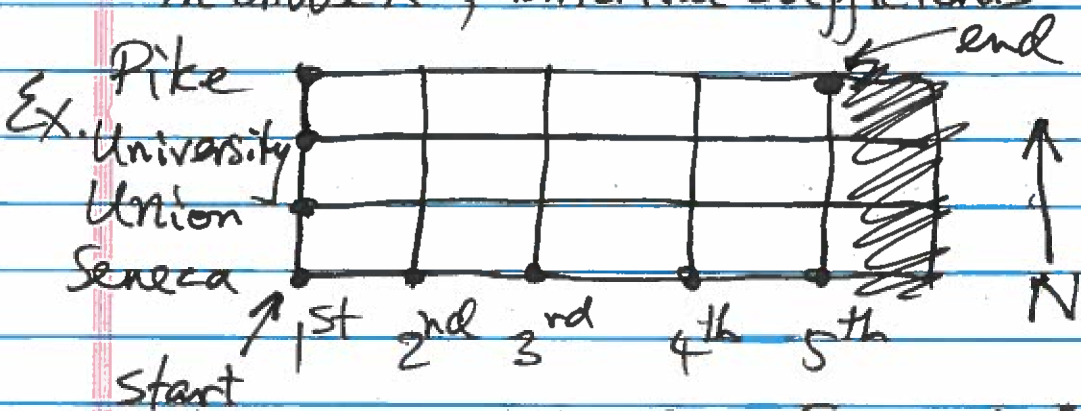
$$\binom{5}{3} = \frac{5\cdot4\cdot3}{3\cdot2\cdot1} = 10$$

Generally, the number of sets of $k$ items chosen from $n$ distinct items is

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

"$n$ choose $k$", binomial coefficients

Ex.




How many routes from Seneca & 1st to Pike & 5th, going either N or E at each intersection?

$\cancel{3!\,4!}$

$$\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7\cdot6\cdot5}{3\cdot2} = 35$$

NNNEEEE

$$= \binom{7}{4}$$

## Identities:

1. $\binom{n}{k} = \binom{n}{n-k}$, for any $0 \le k \le n$

2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, for any $1 \le k \le n-1$.

Either the first object of the n is chosen: $\binom{n-1}{k-1}$ ways to finish

Or it's not chosen: $\binom{n-1}{k}$ ways to finish

Ex: Assume ♣J is the face-up trump. How many starting Schnapsen hands have $\ge 1$ trump?

$$\binom{4}{1}\binom{18}{4} = 12{,}240$$

number of ways to deal 1 trump    number of ways to deal 4 more cards.

This formula is wrong and overcounts!

♣Q   ♣T ♡KQJ

♣T   ♣Q ♡KQJ

How do we find the correct answer? Complementing.

How many starting hands have 0 trumps? $\binom{15}{5}$. How many starting hands in total? $\binom{19}{5}$

Number of hands with $\ge 1$ trump is

$$\binom{19}{5} - \binom{15}{5} = \frac{19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}{5!} - \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{5!}$$

$$= 8625$$

**Binomial Theorem**: for any $n \geq 0$, and for any real numbers $x$ and $y$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Ex**: $n=3$:

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= xxx + xxy + xyx + yxx$$
$$+ xyy + yxy + yyx + yyy$$
$$= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

**Proof for general $n$**:

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n}$$

Collect all the terms of the form $x^k y^{n-k}$. How many such terms are there? The number of ways of choosing $k$ of the $n$ factors to contribute $x$, the rest $y$: $\binom{n}{k}$