CSE 312: Foundations of Computing II
Quiz Section #7: Poisson Distribution, Continuous Random Variables
(solutions)

Review: Main Theorems and Concepts

Poisson: $X \sim Poi(\lambda)$ iff X has the following probability mass function:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$\mathbb{E}[X] = \lambda$ and $\text{Var}(X) = \lambda$. An example of a Poisson r.v. is the number of people born during a particular minute, where λ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson r.v.'s, where $X_i \sim Poi(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim Poi(\lambda_1 + \ldots + \lambda_n)$.

Continuous Random Variable: A r.v. which can take on an uncountably infinite number of values.

Probability Density Function (PDF or density): Let X be a continuous random variable. Then $f_X(x) : \mathbb{R} \to \mathbb{R}$ is a probability density function iff $\forall x\, f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$. **Densities are not probabilities**: in general $f_X(x) \neq \mathbb{P}(X = x)$, since $\mathbb{P}(X = x) = 0$ for all x if X is continuous. Instead, the **area under the curve** is the probability that X falls into a particular range: $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$. The probability that X is close to x is proportional to $f_X(x)$: for small δ,

$\mathbb{P}\left(x - \frac{\delta}{2} < X < x + \frac{\delta}{2}\right) \approx \delta f_X(x)$.

Cumulative Distribution Function (CDF): For a continuous random variable X, $\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt$ and therefore $F_X'(x) = f_X(x)$. In other words, for every possible value (x) that the random variable X can take, the cdf returns the probability that X is less than or equal to that value. Notice that $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$.

Note: Computing the PDF of a new random variable is tricky, because densities are not probabilities. It is usually easier to start by computing the CDF, or $\mathbb{P}(X \leq x)$. Then, take the derivative of the CDF to get the PDF.

Univariate: Discrete to Continuous:

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$E[g(X)] = \sum_x g(x) p_X(x)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

Math formulas used in CSE 312

Definition of exponential:

$$e^x \triangleq \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$$

Geometric series: if $|x| < 1$,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

Zoo of Continuous Random Variables

Uniform: $X \sim Uni(a, b)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$. This represents each real number from $[a, b]$ to be equally likely.

Exponential: $X \sim Exp(\lambda)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$. $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. The exponential random variable is the continuous analog of the geometric random variable: it represents the waiting time to the next event, where $\lambda > 0$ is the average number of events per unit time. Note that the exponential measures how much time passes until the next event (any real number, continuous), whereas the Poisson measures how many events occur in a unit of time (nonnegative integer, discrete). The exponential random variable X is memoryless:

$$\text{for any } s, t \geq 0, \ \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

The geometric random variable also has this property.

Exercises

1. **(Together)** Suppose that $X_1 \sim Poi(\lambda_1)$ and $X_2 \sim Poi(\lambda_2)$. If X_1 and X_2 are independent, show that $Y = X_1 + X_2 \sim Poi(\lambda_1 + \lambda_2)$.

   We find the PMF for Y. For $Y = X_1 + X_2 = n$ to be a particular value, we can have X_1 be 0 and X_2 be n, or X_1 be 1 and X_2 be $n - 1$, or X_1 be 2 and X_2 be $n - 2$, ..., or X_1 be n and X_2 be 0.

$$\begin{aligned}
\mathbb{P}(Y = n) &= \sum_{k=0}^{n} \mathbb{P}(X_1 = k \cap X_2 = n - k) && \text{(as described above)} \\
&= \sum_{k=0}^{n} \mathbb{P}(X_1 = k)\mathbb{P}(X_2 = n - k) && \text{(X_1, X_2 independent)} \\
&= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} && \text{(Plug in Poisson PMF)} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} && \text{(Pull terms out of sum, and multiply both top and bottom by $n!$)} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} && \text{(Definition of $\binom{n}{k}$)} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n && \text{(Binomial Theorem)}
\end{aligned}$$

   This is precisely the PMF for $Poi(\lambda_1 + \lambda_2)$.

2. Alex decided he wanted to create a "new" type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We'll denote a random variable X having the "Uniform-2" distribution as $X \sim Unif2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

   (a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition).

$$f_X(x) = \begin{cases} \frac{1}{(b-a)+(d-c)}, & x \in [a, b] \cup [c, d] \\ 0, & \text{otherwise} \end{cases}$$

   (b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition).

$$F_X(x) = \begin{cases} 0, & x \in (-\infty, a) \\ \frac{(x-a)}{(b-a)+(d-c)}, & x \in [a, b) \\ \frac{(b-a)}{(b-a)+(d-c)}, & x \in [b, c) \\ \frac{(b-a)+(x-c)}{(b-a)+(d-c)}, & x \in [c, d) \\ 1, & x \in [d, \infty) \end{cases}$$

3. You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest square B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For X, find the CDF $F_X(x)$, PDF $f_X(x)$, expectation $\mathbb{E}[X]$, and variance $Var(X)$.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2/s^2, & \text{if } 0 \le x \le s \\ 1, & \text{if } x > s \end{cases}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 2x/s^2, & \text{if } 0 \le x \le s \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_0^s x f_X(x) dx = \int_0^s \frac{2x^2}{s^2} dx = \frac{2}{s^2} \int_0^s x^2 dx = \frac{2}{3s^2} \left[x^3\right]_0^s = \frac{2}{3} s$$

$$\mathbb{E}[X^2] = \int_0^s x^2 f_X(x) dx = \int_0^s \frac{2x^3}{s^2} dx = \frac{2}{s^2} \int_0^s x^3 dx = \frac{1}{2s^2} \left[x^4\right]_0^s = \frac{1}{2} s^2$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{2} s^2 - \left(\frac{2}{3} s\right)^2 = \frac{1}{18} s^2$$

4. Suppose $X \sim Unif(0, 1)$ and $Y = e^X$. Find $f_Y(y)$.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}\left(e^X \leq y\right) = \mathbb{P}(X \leq \ln y) = \begin{cases} 0, & \text{if } y < 1 \\ \ln y, & \text{if } y \in [1, e] \\ 1, & \text{if } y > e \end{cases}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{y}, & \text{if } y \in [1, e] \\ 0, & \text{otherwise} \end{cases}$$

5. Prove the memorylessness property for the exponential distribution $\text{Exp}(\lambda)$: If s and t are nonnegative, then $\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$.

$$\begin{aligned} \mathbb{P}(X > s + t \mid X > s) &= \frac{\mathbb{P}(X > s + t \cap X > s)}{\mathbb{P}(X > s)} \\ &= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= \mathbb{P}(X > t) \end{aligned}$$