CSE 312: Foundations of Computing II
Additional Problems #7: Poisson Distribution, Continuous Random Variables
(solutions)

1. Prove the memorylessness property for the geometric distribution Geo(p).

$$
\begin{aligned}
\mathbb{P}(X > s+t \mid X > s) &= \frac{\mathbb{P}(X > s+t \cap X > s)}{\mathbb{P}(X > s)} \\
&= \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > s)} \\
&= \frac{(1-p)^{s+t}}{(1-p)^s} \\
&= (1-p)^t \\
&= \mathbb{P}(X > t)
\end{aligned}
$$

2. A single-stranded (1-dimensional) spider web, with length W centimeters, where $W > 4$, is stretched taut between two fence posts. The homeowner (a spider) sits precisely at the midpoint of this web. Suppose that a fly gets caught at a random point on the strand, with each point being equally likely.

   (a) The spider is lazy, and it is only willing to walk over and eat the fly if the fly lands within 2 centimeters of where the spider sits. What is the probability that the spider eats the fly?

$$\frac{4}{W}$$

   (b) Let X be the random variable that represents the spider's distance from the fly's landing point. Calculate the CDF, PDF, expectation, and variance of X.

$$F_X(x) = \mathbb{P}(X \le x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{2x}{W} & \text{if } 0 \le x \le \frac{W}{2} \\ 1, & \text{if } x > \frac{W}{2} \end{cases}$$

$$f_X(x) = F'_X(x) = \begin{cases} \frac{2}{W}, & \text{if } 0 \le x \le \frac{W}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_0^{W/2} \frac{2}{W} x\,dx = \frac{2}{W}\left[\frac{1}{2}x^2\right]_0^{\frac{W}{2}} = \frac{W}{4}$$

$$\mathbb{E}\left[X^2\right] = \int_0^{W/2} \frac{2}{W} x^2 dx = \frac{2}{W}\left[\frac{1}{3}x^3\right]_0^{\frac{W}{2}} = \frac{2}{W}\cdot\frac{1}{3}\cdot\frac{W^3}{8} = \frac{W^2}{12}$$

$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] = \frac{W^2}{12} - \frac{W^2}{16} = \frac{W^2}{48}$$

Alternatively, it is clear that $X \sim Uni\left(0, \frac{W}{2}\right)$, so $f_X(x) = \frac{2}{W}, 0 \le x \le \frac{W}{2}$, $F_X(x) = \frac{2x}{W}, 0 \le x \le \frac{W}{2}$ $\mathbb{E}[X] = \frac{W}{4}$ and $\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{\left(\frac{W}{2}\right)^2}{12} = \frac{W^2}{48}$.

3. Starting from the PDF of $X \sim \text{Exp}(\lambda)$, prove that $\mathbb{E}[X] = 1/\lambda$. (Hint: use integration by parts.)

   $\mathbb{E}[X] = \int_0^\infty x\lambda e^{-\lambda x}dx$. Use integration by parts with $u = x$ and $v = -e^{-\lambda x}$, so that $dv = \lambda e^{-\lambda x}dx$. Then

$$\begin{aligned}
\mathbb{E}[X] &= -xe^{-\lambda x}\mid_0^\infty - \int_0^\infty -e^{-\lambda x}dx \\
&= 0 - \frac{1}{\lambda}e^{-\lambda x}\mid_0^\infty \\
&= 1/\lambda
\end{aligned}$$

4. Starting from the probability density function of $X \sim \text{Exp}(\lambda)$, prove that $\mathbb{P}(X \ge t) = e^{-\lambda t}$, for $t \ge 0$. As a corollary, show that the cumulative distribution function for X is $F_X(t) = 1 - e^{-\lambda t}$.

   $\mathbb{P}(X \ge t) = \int_t^\infty \lambda e^{-\lambda x}dx = -e^{-\lambda x}\mid_t^\infty = e^{-\lambda t}$. $F_X(t) = \mathbb{P}(X \le t) = 1 - \mathbb{P}(X \ge t) = 1 - e^{-\lambda t}$.

5. Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f(x) = \frac{c}{1+x^2}$ is valid. Then find $\mathbb{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan\frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

$$\int_0^\infty \frac{c}{1+x^2}dx = c\tan^{-1} x\mid_0^\infty = c\left(\frac{\pi}{2} - 0\right) = 1$$

   so $c = 2/\pi$.

$$\mathbb{E}[X] = \int_0^\infty \frac{cx}{1+x^2}dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}dx = \frac{1}{\pi}\ln(1+x^2)\mid_0^\infty = \infty$$

6. Let $X \sim \text{Exp}(\lambda)$. For $t < \lambda$, find $M_X(t) = \mathbb{E}[e^{tX}]$. M is called the *moment generating function* of X. Find $M'_X(0)$ and $M''_X(0)$. Do you notice any relationship between these two values and $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ (which are sometimes called the first and second *moments* of X)?

$$
\begin{aligned}
M_X(t) &= \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} |_0^\infty = \frac{\lambda}{\lambda - t} \\
M_X'(t) &= \frac{\lambda}{(\lambda - t)^2} \\
M_X'(0) &= \frac{1}{\lambda} = \mathbb{E}[X] \\
M_X''(t) &= \frac{2\lambda}{(\lambda - t)^3} \\
M_X''(0) &= \frac{2}{\lambda^2} \\
\mathbb{E}[X^2] &= \text{Var}(X) + (\mathbb{E}[X])^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = M_X''(0)
\end{aligned}
$$

7. Suppose $X_1, ..., X_n$ are independent with $X_i \sim Exp(\lambda_i)$.

(a) Show that $X = \min(X_1, .., X_n) \sim Exp(\lambda)$, where $\lambda = \sum_{k=1}^n \lambda_k$.

First, $\mathbb{P}(X_i > x) = e^{-\lambda_i x}$ for all i and x. Then, since the min is greater than x iff all its arguments are:

$$\mathbb{P}(X > x) = \mathbb{P}(\min(X_1, ..., X_n) > x) = \mathbb{P}\left(\bigcap_{i=1}^n X_i > x\right) = \prod_{i=1}^n \mathbb{P}(X_i > x) = \prod_{i=1}^n e^{-\lambda_i x} = e^{-\lambda x}$$

So $\mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x) = 1 - e^{-\lambda x}$, so $X \sim Exp(\lambda)$.

(b) Suppose I have a device that needs two batteries at all times. We have n batteries, each of which has lifetime according to $Exp(\lambda)$, independently of other batteries. Assume we instantaneously switch to a new battery when one dies. Initially, we use 2 of the n batteries. What is the expected time I can operate this device?

We start with two batteries; the time for the first of them to die is $Exp(2\lambda)$ by the previous part. So each time I have two batteries, the expected time until the first one fails is $\frac{1}{2\lambda}$. Because the exponential is **memoryless**, when one fails, we can put in another battery, and it will be like starting over. We can replace batteries $n-2$ times, and then we can wait for one of the last two to expire. Hence we have $n-1$ "lifetimes". By linearity of expectation, the expected time we can operate this device is $\frac{n-1}{2\lambda}$.

8. We show a useful lemma for calculating expectation for nonnegative random variables.

(a) Let X be a **nonnegative integer-valued** discrete random variable; that is, one that takes on values in some subset of $\{0, 1, 2, ...\}$. Show that $\mathbb{E}[X] = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$.

We wish to find $\mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}(X = k) = 1 \cdot \mathbb{P}(X = 1) + 2 \cdot \mathbb{P}(X = 2) + 3 \cdot \mathbb{P}(X = 3) + \ldots$

We can write this in a table, and it corresponds to summing row by row (ignoring the first row)

$\mathbb{P}(X > 0)$ ———	$\mathbb{P}(X > 1)$ ———	$\mathbb{P}(X > 2)$ ———	$\mathbb{P}(X > 3)$ ———	$\mathbb{P}(X > 4)$ ———	———
$\mathbb{P}(X = 1)$					
$\mathbb{P}(X = 2)$	$\mathbb{P}(X = 2)$				
$\mathbb{P}(X = 3)$	$\mathbb{P}(X = 3)$	$\mathbb{P}(X = 3)$			
$\mathbb{P}(X = 4)$	$\mathbb{P}(X = 4)$	$\mathbb{P}(X = 4)$	$\mathbb{P}(X = 4)$		
$\mathbb{P}(X = 5)$	$\mathbb{P}(X = 5)$	$\mathbb{P}(X = 5)$	$\mathbb{P}(X = 5)$	$\mathbb{P}(X = 5)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

We get the lemma immediately by considering summing column by column (see the top row for column sums). The first column is precisely $\mathbb{P}(X > 0)$, the second column is $\mathbb{P}(X > 1)$, etc.

(b) Let X be a **nonnegative** continuous random variable; that is, one that takes on values in some subset of $[0, \infty)$. Show that $\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X > x)dx$. (Hint: use double integrals and reverse the order of integration).

$$\int_0^{\infty} \mathbb{P}(X > x)dx = \int_0^{\infty} \int_x^{\infty} f_X(y)dydx = \int_0^{\infty} \int_0^{y} f_X(y)dxdy = \int_0^{\infty} f_X(y) \int_0^{y} dxdy = \int_0^{\infty} y f_X(y)dy = \mathbb{E}[X]$$

(c) Let's recalculate the mean of the geometric random variable using this lemma (avoiding Taylor series).

i. Find $\mathbb{P}(X > k)$ if $X \sim Geo(p)$.

$$\mathbb{P}(X > k) = \mathbb{P}(\text{first } k \text{ all tails}) = (1 - p)^k$$

ii. Rederive the mean of X using part (a).

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} \mathbb{P}(X > k) = \sum_{k=0}^{\infty} (1 - p)^k = \frac{1}{1 - (1 - p)} = \frac{1}{p}$$

(d) Let's recalculate the mean of the exponential random variable using this lemma (avoiding integration by parts).

i. Find $\mathbb{P}(X > x)$ if $X \sim Exp(\lambda)$.

We are given $F_X(x)$ above, so $\mathbb{P}(X > x) = 1 - \mathbb{P}(X \leq x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$

ii. Rederive the mean of X using part (b).

Hence,

$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X > x)dx = \int_0^{\infty} e^{-\lambda x}dx = \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty} = \frac{1}{\lambda}$$

9. If electricity power failures occur according to a Poisson distribution with an average of 3 failures every twenty weeks, calculate the probability that there will be more than one failure during a particular week.

$$X \sim Poi\left(\frac{3}{20}\right)$$

$$\mathbb{P}(X > 1) = 1 - p_X(0) - p_X(1) = 1 - \frac{e^{-0.15}0.15^0}{0!} - \frac{e^{-0.15}0.15}{1!} \approx 0.01$$