CSE 312: Foundations of Computing II
Quiz Section #5: Midterm review (solutions)

Review of Concepts

Expectation (expected value, mean, or average): The expectation of a discrete random variable is defined to be $\mathbb{E}[X] = \sum_x x p_X(x) = \sum_x x \mathbb{P}(X = x)$. The expectation of a function of a discrete random variable $g(X)$ is $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$.

Linearity of Expectation: Let X and Y be random variables, and $a, b, c \in \mathbb{R}$. Then, $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$.

To take advantage of Linearity of Expectation, it is often helpful to write a variable X as a sum of **indicator variables**, which are of the following form:

$$X_i = \begin{cases} 1, & \text{if some condition is met for object } i \\ 0, & \text{otherwise} \end{cases}$$

Then, $\mathbb{E}[X] = \mathbb{E}[\sum_i X_i] = \sum_i \mathbb{E}[X_i] = \sum_i \mathbb{P}(X_i = 1)$.

Variance: Let X be a random variable and $\mu = \mathbb{E}[X]$. The variance of X is defined to be $\mathrm{Var}(X) = \mathbb{E}[(X - \mu)^2]$. Notice that since this is an expectation of a nonnegative random variable ($(X - \mu)^2$), variance is always nonnegative. With some algebra, we can simplify this to $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X]$.

Property of Variance: Let $a, b \in \mathbb{R}$ and X a random variable. Then, $\mathrm{Var}(aX + b) = a^2 Var(X)$.

Exercises on Expectation and Variance

1. Let the random variable X be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

   (a) What is the probability mass function of X?

$$\begin{aligned} p_X(2) &= 1/9 \\ p_X(3) &= 2/9 \\ p_X(4) &= 3/9 \\ p_X(5) &= 2/9 \\ p_X(6) &= 1/9 \end{aligned}$$

(b) Find $\mathbb{E}[X]$ directly by applying the definition of expectation to the result from part (a).

$$\mathbb{E}[X] = 2 \cdot \frac{1}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 5 \cdot \frac{2}{9} + 6 \cdot \frac{1}{9} = 4$$

(c) Find $\mathbb{E}[X]$ again using linearity of expectation.

Let Y be the result of the first roll and Z be the result of the second.

$$\mathbb{E}[X] = \mathbb{E}[Y + Z] = \mathbb{E}[Y] + \mathbb{E}[Z] = 2\mathbb{E}[Y] = 2 \cdot \frac{1}{3}(1 + 2 + 3) = 4$$

(d) Now compute $Var(X)$ two ways: (1) using the definition $Var(X) = \mathbb{E}[(X - \mu)^2]$, and (2) using the formula $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

Method 1: From part (c), the mean (or expected value) of μ is just 4. Now, we apply the formula for the expectation of a function of a random variable.

$$\begin{aligned}
Var(X) = \mathbb{E}((X - 4)^2) &= \sum_x (X - 4)^2 p_X(x) \\
&= (2-4)^2 \cdot p_X(2) + (3-4)^2 \cdot p_X(3) + (4-4)^2 \cdot p_X(4) + (5-4)^2 \cdot p_X(5) + (6-4)^2 \cdot p_X(6) \\
&= (-2)^2 \cdot \frac{1}{9} + (-1)^2 \cdot \frac{2}{9} + 0^2 \cdot \frac{3}{9} + 1^2 \cdot \frac{2}{9} + 2^2 \cdot \frac{1}{9} \\
&= \boxed{\frac{4}{3}}
\end{aligned}$$

As a reminder, $p_X(x)$ means the probability that the random variable X takes on the value x.

Method 2: First we compute $\mathbb{E}[X^2]$, using the formula for expectation of a function of a random variable.

$$\begin{aligned}
E[X^2] &= 2^2 \cdot p_X(2) + 3^2 \cdot p_X(3) + 4^2 \cdot p_X(4) + 5^2 \cdot p_X(5) + 6^2 + \cdot p_X(6) \\
&= 4 \cdot \frac{1}{9} + 9 \cdot \frac{2}{9} + 16 \cdot \frac{3}{9} + 25 \cdot \frac{2}{9} + 36 \cdot \frac{1}{9} \\
&= \frac{52}{3}
\end{aligned}$$

Now apply the formula for variance:

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{52}{3} - 4^2 = \boxed{\frac{4}{3}}$$

2. You are playing a game at a primitive casino. To play, you must pay \$20 initially. Then, you roll one fair 6-sided dice, and you are paid 5 times the value you roll. Let M be the amount of money you earn as profit from playing the game once. Compute $\mathbb{E}[M]$ and $Var(M)$. Use the fact that if X is the value of a single roll of a fair 6-sided dice, $\mathbb{E}[X] = 7/2$ and $Var(X) = 105/36$.

   $M = 5X - 20$, so:

$$\mathbb{E}[M] = \mathbb{E}[5X - 20] = 5\mathbb{E}[X] - 20 = 5 \cdot \frac{7}{2} - 20 = \boxed{-\frac{5}{2}}$$

$$Var(M) = Var(5X - 20) = 5^2 \cdot Var(X) = 25 \cdot \frac{105}{36} = \boxed{\frac{875}{12}}$$

3. You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

   (a) What is the probability mass function of X?

$$\begin{aligned}
p_X(8) &= \frac{\binom{10}{2}}{\binom{20}{4}} = \frac{3}{19 \cdot 17} \\
p_X(7) &= \frac{\binom{10}{1}\binom{9}{2} \cdot 2^2}{\binom{20}{4}} = \frac{96}{19 \cdot 17} \\
p_X(6) &= \frac{\binom{10}{4} \cdot 2^4}{\binom{20}{4}} = \frac{224}{19 \cdot 17} \\
p_X(i) &= 0 \text{ for all other values of } i
\end{aligned}$$

   (b) Find $\mathbb{E}[X]$ directly by applying the definition of expectation to the result from part (a). Give your answer exactly as a simplified fraction.

$$\mathbb{E}[X] = \frac{8 \cdot 3 + 7 \cdot 96 + 6 \cdot 224}{19 \cdot 17} = \frac{120}{19}$$

(c) Find $\mathbb{E}[X]$ again using linearity of expectation. Give your answer exactly as a simplified fraction.

For $1 \le i \le 10$, let

$$\begin{aligned}
X_i &= \begin{cases} 1, & \text{if both socks from pair } i \text{ survive} \\ 0, & \text{otherwise} \end{cases} \\
\mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathbb{E}[X_i] \\
&= \sum_{i=1}^{10} (1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0)) \\
&= \sum_{i=1}^{10} \mathbb{P}(\text{both socks from pair } i \text{ survive}) \\
&= \sum_{i=1}^{10} \frac{\binom{18}{4}}{\binom{20}{4}} = \frac{120}{19}
\end{aligned}$$

4. Find the expected number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.)

   For $1 \le i \le n$, let

$$\begin{aligned}
X_i &= \begin{cases} 1, & \text{if bin } i \text{ is empty} \\ 0, & \text{otherwise} \end{cases} \\
\mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] \\
&= \sum_{i=1}^{n} (1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0)) \\
&= \sum_{i=1}^{n} \mathbb{P}(\text{bin } i \text{ is empty}) \\
&= \sum_{i=1}^{n} \frac{(n-1)^m}{n^m} = \frac{(n-1)^m}{n^{m-1}}
\end{aligned}$$

Midterm Review Exercises (more online!)

5. Let A and B be events in the same sample space that each have nonzero probability. For the following statements, state whether it is always true, always false, or it depends on information not given.

   (a) If A and B are mutually exclusive, then they are independent.

   False

   (b) If A and B are independent, then they are mutually exclusive.

   False

   (c) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are mutually exclusive.

   False

   (d) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are independent.

   Depends whether $\mathbb{P}(A \cap B) = 9/16$

6. How many integers in $\{1, 2, \ldots, 360\}$ are divisible by one or more of the numbers 2, 3, and 5?

   Inclusion-exclusion:

$$\frac{360}{2} + \frac{360}{3} + \frac{360}{5} - \frac{360}{2 \times 3} - \frac{360}{2 \times 5} - \frac{360}{3 \times 5} + \frac{360}{2 \times 3 \times 5} = 180 + 120 + 72 - 60 - 36 - 24 + 12 = 264$$

7. A Schnapsen deck has 4 suits with 5 cards in each suit. Suppose a deck of Schnapsen cards is shuffled well and then dealt into 5 piles of 4 cards each. Let E_i refer to the event that pile i has exactly one spade. Compute the probability $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5)$.

   Imagine the cards are laid out in order. You can think of there being 20 "slots", with 4 groups of 5 slots each. We count the number of ways to place the spades into slots **in order**. For the total number of ways to assign the spades to slots: you have 20 slots for the 1st spade, 19 choices for 2nd spade, and so on. This is the bottom of the fraction (sample space).

   To count the top, we want to make sure that each spade goes to a different pile. There are still 20 possible slots for the 1st spade, but there are only 16 slots for the 2nd spade since it must go into a different pile from the 1st. For the 3rd spade, there are 12 possible slots because it must go into a different pile from the first two, and so on. So we have

$$\frac{20 \cdot 16 \cdot 12 \cdot 8 \cdot 4}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16} \approx 0.066$$

8. You are working on a difficult passage from a new piece you are learning on the piano. You wish to play it correctly 4 times before stopping for the day. If your probability of playing it correctly on each attempt is 2/3, and the attempts are independent (unfortunately!), what is the probability that you have to play it at least 8 times?

   This is equivalent to asking the probability that, in the first 7 attempts, you play it correctly 3 or fewer times. Let X be the number of times you play it correctly in the first 7 attempts. Then $X \sim \mathrm{Bin}(7, 2/3)$.

$$\mathbb{P}(X \le 3) = \binom{7}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^7 + \binom{7}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^6 + \binom{7}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^5 + \binom{7}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4 = \frac{379}{3^7} \approx 0.173$$