CSE 312: Foundations of Computing II
Additional Problems #4: Conditional Probability, Random Variables, Naive Bayes (solutions)

1. Suppose we randomly generate a number from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. Let A_k be the event we generate the number k, and suppose $\mathbb{P}(A_k) = (\frac{1}{2})^k$. Once we generate a number, suppose the probability that we win \$$j$ for $j = 1, \ldots, k$ is "uniform", that is, each has probability $\frac{1}{k}$. Let B be the event we win exactly \$1. What is $\mathbb{P}(A_1|B)$? You may use the fact that $\sum_{j=1}^{\infty} \frac{1}{j \cdot a^j} = \ln(\frac{a}{a-1})$ for $a > 1$.

$$\mathbb{P}(A_1|B) = \frac{\mathbb{P}(B|A_1)\mathbb{P}(A_1)}{\sum_{j=1}^{\infty} \mathbb{P}(B|A_j)\mathbb{P}(A_j)} = \frac{\frac{1}{1}\frac{1}{2^1}}{\sum_{j=1}^{\infty} \frac{1}{j}\frac{1}{2^j}} = \frac{1}{2\ln 2} \approx \boxed{0.7213}$$

2. Suppose you are taking a multiple-choice test that has c answer choices for each question. In answering a question on this test, the probability that you know the correct answer is p. If you don't know the answer, you choose one at random, with each choice equally probable. What is the probability that you knew the correct answer to a question, given that you answered it correctly?

$$\frac{p}{p + (1-p)\frac{1}{c}}$$

3. A couple has 2 children. What is the probability that both are girls, given that the older one is a girl?

   1/2, because the genders of the two children are independent.

4. A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

   (a) If the system is functioning, what is the probability that component 1 is working?

   $$\frac{p}{1-(1-p)^n}$$

   If the system is functioning and component 2 is working, what is the probability that component 1 is working?

   $$p$$

5. A girl has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If she transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

   By the Law of Total Probability,

$$\frac{5}{8} \cdot \frac{5}{9} + \frac{3}{8} \cdot \frac{4}{9} = \frac{37}{72}$$

6. In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

   (a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

   Let E be the event that the drug is effective for Sneezy, and C_i be the event that he gets i colds the first winter. By Bayes' Theorem,

$$\mathbb{P}(E \mid C_1) = \frac{\mathbb{P}(C_1 \mid E)\mathbb{P}(E)}{\mathbb{P}(C_1 \mid E)\mathbb{P}(E) + \mathbb{P}(C_1 \mid \overline{E})\mathbb{P}(\overline{E})} = \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.2 \times 0.8} = \frac{3}{11}$$

   (b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy?

   Let the reduced sample space for part (b) be C_1 from part (a). Let D_i be the event that he gets i colds the second winter. By Bayes' Theorem,

$$\mathbb{P}(E \mid D_2) = \frac{\mathbb{P}(D_2 \mid E)\mathbb{P}(E)}{\mathbb{P}(D_2 \mid E)\mathbb{P}(E) + \mathbb{P}(D_2 \mid \overline{E})\mathbb{P}(\overline{E})} = \frac{0.2 \times \frac{3}{11}}{0.2 \times \frac{3}{11} + 0.2 \times \frac{8}{11}} = \frac{3}{11}$$

(c) The third winter he decides not to bother taking the drug and gets 2 colds. He argues that the drug must not have been effective for him, since he got the same number of colds last year as this year. Comment on his logic.

The posterior probability that the drug is effective is 3/11. This is greater than the prior probability 1/5, so the drug probably was effective.

7. Suppose we have N items in a bag, K of which are successes. Suppose we draw (without replacement) until we have k successes, $k \le K \le N$. Let X be the number of draws until the k^{th} success. What is Ω_X? What is $p_X(n) = \mathbb{P}(X = n)$? (We say X is a "negative hypergeometric" random variable).

$$\Omega_X = \{k, k+1, \dots N - K + k\}$$

$$p_X(n) = \mathbb{P}(X = n) = \frac{\binom{K}{k-1}\binom{N-K}{n-k}}{\binom{N}{n-1}} \frac{K - (k-1)}{N - (n-1)}, \quad n = k, k+1, \dots, N - K + k$$

8. Guildenstern has a fair coin and a "magic" coin that comes up heads with probability $p_1 > \frac{1}{2}$. Suppose he picks a coin at random, with probability p_2 of choosing the magic coin and $1 - p_2$ of choosing the fair coin, and tosses it n times. All of the tosses come up heads. He would like to convince Rosencrantz that he flipped the magic coin. Rosencrantz only believes him if the conditional probability that it is the magic coin, given the n heads, is at least 99%. Derive a function $n = f(p_1, p_2)$ that gives the minimum number of consecutive heads n to convince Rosencrantz that Guildenstern flipped the magic coin. Remember that n must be a positive integer.

Let M be the event that Guildenstern picked the magic coin, and H be the event that he flipped n heads in a row. By Bayes' Theorem,

$$\begin{aligned}
\mathbb{P}(M \mid H) &= \frac{\mathbb{P}(H \mid M)\mathbb{P}(M)}{\mathbb{P}(H \mid M)\mathbb{P}(M) + \mathbb{P}(H \mid \overline{M})\mathbb{P}(\overline{M})} \\
&= \frac{p_1^n p_2}{p_1^n p_2 + (\frac{1}{2})^n(1-p_2)} \\
&\geq 0.99 \\
0.01 p_1^n p_2 &\geq 0.99(1-p_2)/2^n \\
(2p_1)^n &\geq 99 \cdot \frac{1-p_2}{p_2} = \frac{99}{p_2} - 99 \\
n &= \left\lceil \frac{\log(\frac{99}{p_2} - 99)}{\log(2p_1)} \right\rceil
\end{aligned}$$

9. This problem demonstrates that independence can be "broken" by conditioning. Let D_1 and D_2 be the outcomes of two independent rolls of a fair die. Let E be the event "$D_1 = 1$", F be the event "$D_2 = 6$", and G be the event "$D_1 + D_2 = 7$". Even though E and F are independent, show that

$$\mathbb{P}(E \cap F \mid G) \neq \mathbb{P}(E \mid G)\, \mathbb{P}(F \mid G).$$

$$\begin{aligned}
\mathbb{P}(E \mid G) &= \mathbb{P}(D_1 = 1 \mid D_1 + D_2 = 7) = 1/6 \\
\mathbb{P}(F \mid G) &= \mathbb{P}(D_2 = 6 \mid D_1 + D_2 = 7) = 1/6 \\
\mathbb{P}(E \cap F \mid G) &= \mathbb{P}(D_1 = 1 \cap D_2 = 6 \mid D_1 + D_2 = 7) = 1/6
\end{aligned}$$

10. Guildenstern has three coins C_1, C_2, C_3 in a bag. C_1 has P(heads) = 1, C_2 has P(heads) = 0, and C_3 has P(heads) = p. He takes a random coin from the bag, each coin equally probable, and flips this same coin some number of times.

    (a) Suppose q is the conditional probability that he flipped coin C_1, given that the flip came up heads. Determine p as a function of q.

    Let F_i be the event that he flipped coin C_i and H be the event that the flip came up heads. By Bayes' Theorem,

$$\begin{aligned}
q &= \mathbb{P}(F_1 \mid H) = \frac{\mathbb{P}(H \mid F_1)\mathbb{P}(F_1)}{\mathbb{P}(H \mid F_1)\mathbb{P}(F_1) + \mathbb{P}(H \mid F_2)\mathbb{P}(F_2) + \mathbb{P}(H \mid F_3)\mathbb{P}(F_3)} \\
&= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 \times \frac{1}{3} + p \times \frac{1}{3}} \\
&= \frac{1}{1+p} \\
p &= \frac{1}{q} - 1
\end{aligned}$$

(b) What is the probability that the first n flips come up tails?

Let T be the event that he flips n tails in a row. By the Law of Total Probability,

$$\begin{aligned}
\mathbb{P}(T) &= \mathbb{P}(T \mid F_1)\mathbb{P}(F_1) + \mathbb{P}(T \mid F_2)\mathbb{P}(F_2) + \mathbb{P}(T \mid F_3)\mathbb{P}(F_3) \\
&= 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + (1-p)^n \times \frac{1}{3} \\
&= \frac{1}{3}(1 + (1-p)^n)
\end{aligned}$$

(c) Given that the first n flips come up tails, what is the probability he flipped C_1? C_2? C_3?

By Bayes' Theorem,

$$
\begin{aligned}
\mathbb{P}(F_1 \mid T) &= \frac{\mathbb{P}(T \mid F_1)\mathbb{P}(F_1)}{\mathbb{P}(T)} \\
&= \frac{0 \times \frac{1}{3}}{\frac{1}{3}(1 + (1-p)^n)} \\
&= 0 \\
\mathbb{P}(F_2 \mid T) &= \frac{\mathbb{P}(T \mid F_2)\mathbb{P}(F_2)}{\mathbb{P}(T)} \\
&= \frac{1 \times \frac{1}{3}}{\frac{1}{3}(1 + (1-p)^n)} \\
&= \frac{1}{1 + (1-p)^n} \\
\mathbb{P}(F_3 \mid T) &= \frac{\mathbb{P}(T \mid F_3)\mathbb{P}(F_3)}{\mathbb{P}(T)} \\
&= \frac{(1-p)^n \times \frac{1}{3}}{\frac{1}{3}(1 + (1-p)^n)} \\
&= \frac{(1-p)^n}{1 + (1-p)^n}
\end{aligned}
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