CSE 312: Foundations of Computing II
Quiz Section #3: Pigeonhole Principle, Introduction to Probability
(solutions)

Review: Main Theorems and Concepts

(Strong) Pigeonhole Principle: If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$) pigeons.
Sample Space: The set of all possible outcomes of an experiment, denoted Ω or S
Event: Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
Union: The union of two events E and F is denoted $E \cup F$
Intersection: The intersection of two events E and F is denoted $E \cap F$ or EF
Mutually Exclusive: Events E and F are mutually exclusive iff $E \cap F = \emptyset$
Complement: The complement of an event E is denoted E^C or $\bar{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
DeMorgan's Laws: $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
Probability of an event E: denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$
Partition: Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff

- $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): A and A^C partition Ω

Axioms of Probability and their Consequences

1. **(Non-negativity)** For any event E, $\mathbb{P}(E) \geq 0$
2. **(Normalization)** $\mathbb{P}(\Omega) = 1$
3. **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

- $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
- If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

Equally Likely Outcomes: If we have equally likely outcomes in finite sample space Ω, and E is an event, then $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

Exercises

1. **(From previous worksheet)** At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

   For $i = 1, ..., n$, let r_i be the number of rotations clockwise needed for the i^{th} person to be in their spot. Each r_i can be between 1 and $n - 1$ (not 0 since no one is at their nametag, and not n since it is equivalent to 0). Since there are n people and only $n - 1$ possible values for the rotations, at least two must have the same value by the pigeonhole principle. Rotate the table clockwise by that much, and at least two people will be in the correct place.

2. Show that in any group of n people, there are two who have an identical number of friends within the group. (Assume that friendships are bidirectional.) Let's prove this in cases:

   (a) Case 1: each person has at least 1 friend in the group

   Each individual can have somewhere between 1 to $n-1$ friends; as there are n individuals, by pigeonhole there must be at least two with the same number of friends.

   (b) Case 2: a person has 0 friends in the group

   If there are multiple people with 0 friends, those people have the same number of friends (0), so the condition is proved. If there is exactly 1 person with 0 friends, then the $n-1$ other people must have between 1 to $n-2$ friends. By pigeonhole again, there must be at least two remaining people with the same number of friends.

3. Show that for every $n \geq 2$, $\exists\, k \in \mathbb{N}$ such that $n \mid k$, and has all digits only 1 and 0. **Hint:** Consider the sequence $1, 11, 111, \ldots,$ of length $n + 1$, and think about remainders.

   Consider the sequence $1, 11, 111, \ldots,$ of length $n + 1$. Consider these numbers mod n: there are n possible remainders. By the pigeonhole principle, two of them (a, b) with $a \leq b$ have the same remainder when divided by n. Then, $n|b - a$, and $b - a$ consists of 1s' followed by 0's.

   If you don't remember why a, b having the same remainder mod n implies their difference is divisible by n, consider the following. By the fundamental theorem of arithmetic, $a = d_1 n + r$

and $b = d_2 n + r$. So $b - a = n(d_2 - d_1)$, where $d_2 - d_1$ is an integer. Hence $b - a$ is divisible by n.

4. An urn contains 3 black balls and 4 white balls.

   (a) Suppose 3 balls are drawn from the urn without replacement. What is the probability that all 3 are white? Try computing this in the sample space where the order of the 3 draws does not matter, and then in the sample space where the order does matter.

   When order does not matter:

$$\frac{\binom{4}{3}}{\binom{7}{3}} = \frac{4 \cdot 3!}{7 \cdot 6 \cdot 5} = \frac{4}{35} \approx 0.114$$

   When order does matter:

$$\frac{4!/(4-3)!}{7!/(7-3)!} = \frac{4 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5} = \frac{4}{35} \approx 0.114$$

   (b) Suppose 3 balls are drawn from the urn with replacement. What is the probability that all 3 are white? Describe the sample space precisely.

   The sample space consists of all ways of drawing 3 balls with replacement, where the order of the 3 draws matters. The probability is

$$\frac{4^3}{7^3} = \left(\frac{4}{7}\right)^3 \approx 0.187$$

5. Suppose there are N items in a bag, with K of them marked as successes in total (and the rest are marked as failures). We draw n of them, without replacement. Each item is equally likely to be drawn. Let X be the number of successes we draw (out of n). What is $\mathbb{P}(X = k)$, that is, the probability we draw exactly k successes?

$$\boxed{\mathbb{P}(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

   We choose k out of the K successes, and $n - k$ out of the $N - K$ failures. The denominator is the total number of ways to choose n objects out of N.

6. Suppose we have 12 chairs (in a row) with 9 TA's, and Professors Ruzzo, Karlin, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

   Imagine we permute all 9 TA's first – there are 9! ways to do this. Then, there are 8 spots between them, in which we pick 3 for the Professors to sit – order matters since each Professor is distinct. So the total ways is $P(8,3) \cdot 9!$.

   The total number of ways to seat all 12 of us is simply 12!.

   The probability is then $\boxed{\dfrac{P(8,3) \cdot 9!}{12!}}$

7. Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

   (a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

   $$\frac{\binom{14}{10}}{\binom{20}{10}}$$

   (b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 9 Republicans and 1 Democrat?

   $$\frac{\binom{14}{9}\binom{6}{1}}{\binom{20}{10}}$$