CSE 312: Foundations of Computing II
Additional Problems #3: Introduction to Probability (solutions)

1. Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

   First, note there are $2k$ objects – k shoes and k socks. Suppose we describe a sequence of actions, $Sock_1, Shoe_1, Sock_2, Shoe_2, \ldots, Sock_k, Shoe_k$.

   This particular example means we first put a sock on leg 1, then a shoe on leg 1, then a sock on leg 2, etc. There are $(2k)!$ ways to order these actions. However, for each leg, there is only one valid ordering: the sock must come before the shoe. So we divide by 2^k and the total number of ways is $\frac{(2k)!}{2^k}$.

   Alternatively, $\mathbb{P}(\text{valid ordering}) = \frac{|\text{valid orderings}|}{|\text{orderings}|}$, so $|\text{valid orderings}| = \mathbb{P}(\text{valid ordering}) * |\text{orderings}|$. We can compute $\mathbb{P}(\text{valid ordering}) = (1/2)^k$. Notice for any sequence of actions with each equally likely, the probability that the sock came before the shoe on a particular leg is $\frac{1}{2}$, so the probability this happened for each leg is $(1/2)^k$. Then $|\text{orderings}| = (2k)!$ because there are $2k$ actions that we can permute, all distinct. Multiplication gives the same answer as above.

2. Given 3 different spades and 3 different hearts, shuffle them. Compute $\mathbb{P}(E)$, where E is the event that the suits of the shuffled cards are in alternating order. What is your sample space?

   The sample space is the set of all possible orderings of the 6 cards.

$$\mathbb{P}(E) = \frac{6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1}{6!} = \frac{1}{10}$$

3. Novice poker players are often confused about which player wins if one holds a flush and one holds a straight. For draw poker (see quiz section #1 worksheet, exercise #25):

   (a) Compute the probability of being dealt a flush.

$$\frac{4\binom{13}{5}}{\binom{52}{5}} \approx 0.00198$$

(b) Compute the probability of being dealt a straight.

$$\frac{10 \cdot 4^5}{\binom{52}{5}} \approx 0.00394$$

(c) Which of these hands should win, given your answers to (a) and (b)?

A flush should beat a straight, since it is rarer.

4. Suppose you deal 13 cards from a well-shuffled bridge deck (4 suits with 13 cards in each). What is the probability that the distribution of suits is 4, 4, 3, 2? (That is, you have 4 cards of one suit, 4 cards of another suit, 3 cards of another suit, and 2 cards of the last suit.)

$$\frac{\binom{13}{4}\binom{13}{4}\binom{13}{3}\binom{13}{2} \cdot 4!}{\binom{52}{13} \cdot 2!}$$

The factor of 4! in the numerator takes care of the number of ways to assign suits to the number of cards, and the factor of 2! in the denominator takes care of the fact that two suits have the same number (4) of cards and so are overcounted.

5. (Challenge problem) n people at a reception give their hats to a hat-check person. When they leave, the hat-check person gives each of them a hat chosen at random. What is the probability that no one gets their own hat back?

See slides 37-42 in the third set of lecture slides on the Spring 2015 CSE 312 web, `http://courses.cs.washington.edu/courses/cse312/15sp/slides/03dprob.pdf`.