CSE 312: Foundations of Computing II
Quiz Section #3: Pigeonhole Principle, Introduction to Probability

Review: Main Theorems and Concepts

(Strong) Pigeonhole Principle: If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$) pigeons.
Sample Space: The set of all possible outcomes of an experiment, denoted Ω or S
Event: Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
Union: The union of two events E and F is denoted $E \cup F$
Intersection: The intersection of two events E and F is denoted $E \cap F$ or EF
Mutually Exclusive: Events E and F are mutually exclusive iff $E \cap F = \emptyset$
Complement: The complement of an event E is denoted E^C or $\bar{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
DeMorgan's Laws: $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
Probability of an event E: denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$
Partition: Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff

- $E_1, \ldots, E_n$ are exhaustive: __________________________, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: __________________________
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): ________ and ________ partition Ω

Axioms of Probability and their Consequences

1. **(Non-negativity)** For any event E, $\mathbb{P}(E) \geq$ ________
2. **(Normalization)** $\mathbb{P}(\Omega) =$ ________
3. **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) =$ ________

- $\mathbb{P}(E) + \mathbb{P}(E^C) =$ ________
- If $E \subseteq F$, $\mathbb{P}(E)$ ________ $\mathbb{P}(F)$
- $\mathbb{P}(E \cup F) =$ __________________________

Equally Likely Outcomes: If we have equally likely outcomes in finite sample space Ω, and E is an event, then $\mathbb{P}(E) =$__________________________.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

Exercises

1. **(From previous worksheet)** At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

2. Show that in any group of n people, there are two who have an identical number of friends within the group. (Assume that friendships are bidirectional.) Let's prove this in cases:

   (a) Case 1: each person has at least 1 friend in the group

   (b) Case 2: a person has 0 friends in the group

3. Show that for every $n \geq 2$, $\exists\, k \in \mathbb{N}$ such that $n \mid k$, and has all digits only 1 and 0. **Hint:** Consider the sequence $1, 11, 111, \ldots,$ of length $n + 1$, and think about remainders.

4. An urn contains 3 black balls and 4 white balls.

   (a) Suppose 3 balls are drawn from the urn without replacement. What is the probability that all 3 are white? Try computing this in the sample space where the order of the 3 draws does not matter, and then in the sample space where the order does matter.

   (b) Suppose 3 balls are drawn from the urn with replacement. What is the probability that all 3 are white? Describe the sample space precisely.

5. Suppose there are N items in a bag, with K of them marked as successes in total (and the rest are marked as failures). We draw n of them, without replacement. Each item is equally likely to be drawn. Let X be the number of successes we draw (out of n). What is $\mathbb{P}(X = k)$, that is, the probability we draw exactly k successes?

6. Suppose we have 12 chairs (in a row) with 9 TA's, and Professors Ruzzo, Karlin, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

7. Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

   (a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

   (b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 9 Republicans and 1 Democrat?