CSE 312: Foundations of Computing II
Quiz Section #2: Combinations, Counting Tricks (solutions)

Review: Main Theorems and Concepts

Combinations (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n,k)$$

Multinomial coefficients: Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is:

$$\frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$$

Binomial Theorem: $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Principle of Inclusion-Exclusion (PIE): 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
In general: +singles - doubles + triples - quads + ...

Pigeonhole Principle: If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$) pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

Exercises

1. There are 12 points on a plane. Five of them are collinear and, other than these, no three are collinear.

(a) How many lines, each containing at least 2 of the 12 points, can be formed?

$\boxed{\binom{12}{2} - \binom{5}{2} + 1 = 57}$

(b) How many triangles, each containing at least 3 of the 12 points, can be formed?

$\boxed{\binom{12}{3} - \binom{5}{3} = 210}$

2. There are 6 women and 7 men in a ballroom dancing class. If 4 men and 4 women are chosen and paired off, how many pairings are possible?

   First choose 4 men and 4 women, $\binom{7}{4}$ and $\binom{6}{4}$ respectively. Then, fix the order of men M_1, M_2, M_3, M_4.

   You can see that there are 4! ways to assign the women to each man, so we have a total of $\boxed{\binom{7}{4}\binom{6}{4}4!}$ ways.

3. You have 12 red beads, 16 green beads, and 20 blue beads. How many distinguishable ways are there to place the beads on a string, assuming that beads of the same color are indistinguishable? (The string has a loose end and a tied end, so that reversing the order of the beads gives a different arrangement, unless the pattern of colors happens to form a palindrome.) Try solving the problem two different ways, once using permutations and once using using combinations.

   Using permutations:

$$\frac{48!}{12! \cdot 16! \cdot 20!}$$

   Using combinations:

$$\binom{48}{12}\binom{36}{16} = \frac{48!}{12! \cdot 36!} \cdot \frac{36!}{16! \cdot 20!}$$

4. How many bridge hands have a suit distribution of 5, 5, 2, 1? (That is, you are playing with a standard 52-card deck and you have 5 cards of one suit, 5 cards of another suit, 2 of another suit, and 1 of the last suit.)

   $\boxed{\binom{13}{5}\binom{13}{5}\binom{13}{2}\binom{13}{1} \cdot \frac{4!}{2!}}$: the factor of 4! in the numerator takes care of the number of ways to assign suits to the number of cards, and the factor of 2! in the denominator takes care of the fact that two suits have the same number (5) of cards and so are overcounted.

5. Give a **combinatorial** proof that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Do not use the binomial theorem. (Hint: you can count the number of subsets of $[n] = \{1, 2, \ldots, n\}$). Note: A combinatorial proof is one

in which you explain how to count something in two different ways – then those formulae must be equivalent if they both indeed count the same thing.

Fix a subset of $[n]$ of size k. There are $\binom{n}{k}$ such subsets because we choose any k elements out of the n, with order not mattering since these are sets. Subsets can be of size $k = 0, 1, \ldots, n$. So the total number of subsets of $[n]$ is $\sum_{k=0}^{n} \binom{n}{k}$. On the other hand, each element of $[n]$ is either in a subset or not. So there are 2 possibilities for the first element (in or out), 2 for the second, etc. So there are 2^n subsets of $[n]$. Therefore, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Note that this agrees with the binomial theorem using $x = y = 1$.

6. Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent. Repeat the question for the letters "AAAAABBB".

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \setminus (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \dfrac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \dfrac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \dfrac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\dfrac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \dfrac{9!}{2!2!} - \binom{3}{2} \cdot \dfrac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

For the second question, note that no A's and no B's can be adjacent. So let us put the B's down first: $\_B\_B\_B\_$

By the pigeonhole principle, two A's must go in the same slot, but then they would be adjacent, so there are $\boxed{\text{no ways}}$.

7. At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

$\boxed{N = 11}$: There are $\binom{4}{2}$ combinations of 2 different suits, plus 4 possibilities of having 2 cards of the same suit. With $N = 11$ you can apply the pigeonhole principle.

8. At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

   For $i = 1, ..., n$, let r_i be the number of rotations clockwise needed for the i^{th} person to be in their spot. Each r_i can be between 1 and $n - 1$ (not 0 since no one is at their nametag, and not n since it is equivalent to 0). Since there are n people and only $n - 1$ possible values for the rotations, at least two must have the same value by the pigeonhole principle. Rotate the table clockwise by that much, and at least two people will be in the correct place.

9. (a) Two parents only have 3 bedrooms for their 13 children. If each child is assigned to a bedroom, one of the bedrooms must have at least c children. What is the maximum value of c that makes this statement true? Prove it.

   $c = 5$. Prove that $c > 4$ by contradiction. Suppose not. Then, all bedrooms have ≤ 4 children, so there are ≤ 12 children, a contradiction. Hence at least one bedroom has at least 5 children.

   (b) (Strong Pigeonhole Principle) More generally, what can you say about n children in k bedrooms? Find a general formula for the maximum value of c that guarantees one of the bedrooms must have at least c children.

   $c = \lceil n/k \rceil$. Note that the ordinary Pigeonhole Principle is the special case when $k = n - 1$.

10. Suppose 250 new majors entered the CSE program this fall. There are 200 new majors in CSE 311, 40 in CSE 331, and 150 in CSE 351. Furthermore, 20 new majors are in both CSE 311 and CSE 331, 120 new majors are in both CSE 311 and CSE 351, and 10 new majors are in both CSE 331 and CSE 351. Finally, there are 4 new majors in all three (CSE 311, CSE 331, and CSE 351). How many CSE students are **not in** any of those 3 courses? *(Note: These numbers were made up.)*

    Let A be the set of students in CSE 311, B be the set of students in CSE 331, and C be the set of students in CSE 351. Start by counting the complement, which is the number of students who are in at least one of those three classes, which is $|A \cup B \cup C|$. By the principle of inclusion-exclusion,

$$
\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 200 + 40 + 150 - 20 - 120 - 10 + 4 = 244
\end{aligned}
$$

Finally, the number of new majors who are **not** in one of those three classes is just the total number of new majors minus those who are in at least one of those classes, which is $250 - 244 = \boxed{6}$.

11. Suppose Anna, Bob, Carol, Daniel, and Evelyn are sitting down to eat, and Anna and Bob **must** sit next to each other. How many arrangements are possible if

    (a) They sit in a line

    Since Anna and Bob must sit next to each other, treat them as a single entity. There are $4!$ arrangements. Then, there are 2 ways to arrange Anna and Bob (either Anna sits to the left of Bob, or to the right). $\boxed{2 * 4! = 48}$

    Alternatively, use complementary counting. There are $5!$ total arrangements. Now subtract the number of ways where Anna and Bob are **not** adjacent. If Anna is sitting at an end of the line (2 positions), there are 3 positions for Bob which are not adjacent to Anna. Then, there are $3!$ ways to arrange the remaining people, so there are $2 \cdot 3 \cdot 3!$ arrangements in this case. However, if Anna is not sitting at an end of the line (3 positions), there are 2 positions for Bob which are not adjacent to Anna. Then there are $3!$ ways to arrange the remaining people, so there are $3 \cdot 2 \cdot 3!$ arrangements in this case. Putting this together, we have $5! - (2 \cdot 3 \cdot 3! + 3 \cdot 2 \cdot 3!) = \boxed{48}$

    (b) They are sitting at a circular table (two arrangements are considered equivalent if one can be rotated to give another)

    Again, we treat Anna and Bob as a single entity. So we're arranging 4 people around a circle. If they were in a line, we would have $4!$ arrangements, but this overcounts by a factor of 4 since each of the 4 rotations of the circle were counted separately, but are actually considered the same arrangement. Thus, there are $4!/4$ ways to arrange 4 objects arond the circle. Finally, there are 2 ways to arrange Anna and Bob again. $\boxed{2 \cdot \frac{4!}{4} = 12}$