CSE 312: Foundations of Computing II
Additional Exercises #2: Combinations, Counting Tricks
(solutions)

Note: These exercises are strictly for your own benefit, in case you need extra practice for exams and homework.

Several exercises below deal with a "standard" 52-card deck, such as is used in the games of bridge and poker. This deck consists of 52 cards divided into 4 suits of 13 cards each. The 4 suits are (black) spades ♠, (red) hearts ♡, (black) clubs ♣, and (red) diamonds ♢. The 13 cards ("ranks") of each suit are 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A.

1. This is another poker exercise. Find the minimum number of cards to be dealt to you from a standard 52-card deck to guarantee that you have some 5 cards among them that form . . .

   (a) one pair? (This occurs when the cards have ranks a, a, b, c, d, where a, b, c, and d are all distinct. The suits do not matter.)

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   (b) two pairs? (This occurs when the cards have ranks a, a, b, b, c, where a, b, and c are all distinct. The suits do not matter.)

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   (c) a full house? (This occurs when the cards have ranks a, a, a, b, b, where a and b are distinct. The suits do not matter.)

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   (d) a straight? (A hand is said to form a straight if the ranks of all 5 cards form an incrementing sequence. The suits do not matter. The lowest straight is A, 2, 3, 4, 5 and the highest straight is 10, J, Q, K, A.)

   45

   (e) a flush? (A hand is said to form a flush if all 5 cards are from the same suit.)

   17

   (f) a straight flush (5 cards of the same suit that form a straight)?

   45

2. 25 fleas sit on a 5×5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

   There are two colors on a checkerboard; so 13 are of one color, and 12 are of another. The 13 fleas must jump to the opposite color of which there are only 12 positions, so at least two fleas must land on the same square by the pigeonhole principle.

3. How many functions $f : A \to B$ with $|A| = a \geq |B| = b$ are surjective? *It's fine if the formula is messy; it will involve a summation.*

   Let $\mathcal{F}$ be the set of all functions $f : A \to B$. Let $B = \{x_1, ..., x_b\}$. Let $\mathcal{F}_i \subseteq \mathcal{F}$ be the set of functions that do *not* hit x_i. These are the ones we are trying to avoid. We want $|\mathcal{F} \setminus (\bigcup_{i=1}^{b} \mathcal{F}_i)| = |\mathcal{F}| - |\bigcup_{i=1}^{b} \mathcal{F}_i|$.

   We saw that $|\mathcal{F}| = b^a$. The size of $\mathcal{F}_i$ is $(b-1)^a$. The size of $\mathcal{F}_i \cap \mathcal{F}_j$ is $(b-2)^a$. As you can see, $\bigcap_{i=1}^{k} \mathcal{F}_i$ has size $(b-k)^a$.

   Inclusion exclusion says we need to add singles, subtract doubles, add triples, ... For a subset of B of size k, there are $\binom{b}{k}$ such subsets. For example, there are $\binom{b}{1}$ single subsets $(|\mathcal{F}_1|, |\mathcal{F}_2|, \ldots)$, $\binom{b}{2}$ double subsets $(|\mathcal{F}_1 \cap \mathcal{F}_2|, |\mathcal{F}_1 \cap \mathcal{F}_3|, |\mathcal{F}_1 \cap \mathcal{F}_4|, \ldots)$, and so on. So we have

   $$\binom{b}{1}(b-1)^a - \binom{b}{2}(b-2)^a + \binom{b}{3}(b-3)^a - \binom{b}{4}(b-4)^a \ldots$$

   functions that do not hit at least one value.

   We can write this using a summation. (Try expanding the sum for $a = 3$ to verify that it is correct. The power of -1 is there to get the signs right.)

   $$\left|\bigcup_{i=1}^{b} \mathcal{F}_i\right| = \sum_{k=1}^{a} (-1)^{k+1} \binom{b}{k} (b-k)^a$$

   .

   Now, we need to subtract our expression from b^a to get

$$
\begin{aligned}
|\mathcal{F} \setminus (\bigcup_{i=1}^{b} \mathcal{F}_i)| &= |\mathcal{F}| - \left|\bigcup_{i=1}^{b} \mathcal{F}_i\right| \\
&= b^a - \sum_{k=1}^{b} (-1)^{k+1} \binom{b}{k} (b-k)^a \\
&= \sum_{k=0}^{b} (-1)^k \binom{b}{k} (b-k)^a
\end{aligned}
$$

(If you're confused about the last equality: we first divided $(-1)^{k+1}$ by -1 to get rid of the negative sign before the sum, resulting in $(-1)^k$. Then, note that if you plug in $k = 0$ into the expression inside the sum, it evaluates to b^a. So we can get rid of the separate b^a term, and make it the $(k = 0)$ term of the sum.)