CSE 312: Foundations of Computing II
Quiz Section #2: Combinations, Counting Tricks

Review: Main Theorems and Concepts

Combinations (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

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Multinomial coefficients: Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is:

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Binomial Theorem: _______________________________________

Principle of Inclusion-Exclusion (PIE): 2 events: $|A \cup B| =$ ________________________
3 events: $|A \cup B \cup C| =$ ________________________
In general: ________________________

Pigeonhole Principle: If there are n pigeons with k holes and $n > k$, then at least one hole contains at least ________________________ pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can: _______________________________________

Exercises

1. There are 12 points on a plane. Five of them are collinear and, other than these, no three are collinear.

   (a) How many lines, each containing at least 2 of the 12 points, can be formed?

   (b) How many triangles, each containing at least 3 of the 12 points, can be formed?

2. There are 6 women and 7 men in a ballroom dancing class. If 4 men and 4 women are chosen and paired off, how many pairings are possible?

3. You have 12 red beads, 16 green beads, and 20 blue beads. How many distinguishable ways are there to place the beads on a string, assuming that beads of the same color are indistinguishable? (The string has a loose end and a tied end, so that reversing the order of the beads gives a different arrangement, unless the pattern of colors happens to form a palindrome.) Try solving the problem two different ways, once using permutations and once using using combinations.

4. How many bridge hands have a suit distribution of 5, 5, 2, 1? (That is, you are playing with a standard 52-card deck and you have 5 cards of one suit, 5 cards of another suit, 2 of another suit, and 1 of the last suit.)

5. Give a **combinatorial** proof that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Do not use the binomial theorem. (Hint: you can count the number of subsets of $[n] = \{1, 2, \ldots, n\}$). Note: A combinatorial proof is one in which you explain how to count something in two different ways – then those formulae must be equivalent if they both indeed count the same thing.

6. Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent. Repeat the question for the letters "AAAAABBB".

7. At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

8. At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

9. (a) Two parents only have 3 bedrooms for their 13 children. If each child is assigned to a bedroom, one of the bedrooms must have at least c children. What is the maximum value of c that makes this statement true? Prove it.

   (b) (Strong Pigeonhole Principle) More generally, what can you say about n children in k bedrooms? Find a general formula for the maximum value of c that guarantees one of the bedrooms must have at least c children.

10. Suppose 250 new majors entered the CSE program this fall. There are 200 new majors in CSE 311, 40 in CSE 331, and 150 in CSE 351. Furthermore, 20 new majors are in both CSE 311 and CSE 331, 120 new majors are in both CSE 311 and CSE 351, and 10 new majors are in both CSE 331 and CSE 351. Finally, there are 4 new majors in all three (CSE 311, CSE 331, and CSE 351). How many CSE students are **not in** any of those 3 courses? *(Note: These numbers were made up.)*

11. Suppose Anna, Bob, Carol, Daniel, and Evelyn are sitting down to eat, and Anna and Bob **must** sit next to each other. How many arrangements are possible if

    (a) They sit in a line

    (b) They are sitting at a circular table (two arrangements are considered equivalent if one can be rotated to give another)