CSE 312: Foundations of Computing II
Quiz Section #1: Sets, Functions, Counting (solutions)

Note: The exercises marked as **(not covered yet)** use combinations, which have not been discussed in lecture yet as of Friday, Jan 5. They will not appear on Homework 1.

Note: If you would like more practice for exams and homework, there is another file on the website containing additional problems.

Review: Main Theorems and Concepts

1. **Sets:** A **set** is a collection of distinct objects, called elements. Some important sets include:
   (a) The integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
   (b) The naturals $\mathbb{N} = \{1, 2, 3, ...\}$
   (c) The rationals $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$
   (d) The reals $\mathbb{R}$
   (e) The first n naturals $[n] = \{1, 2, ..., n\}$
2. **Sequences:** A **sequence** is an *ordered* list of elements of a set (can have duplicates). For example, $1, -3, 2, -4, 2, -3$ is a sequence.
3. **Functions:** Let A, B be sets, called the **domain**, and **codomain**, respectively. A **function (or map)** $f : A \to B$ assigns each $a \in A$ to a unique value $b \in B$. We denote the value f maps a to as $f(a) \in B$.
   (a) The **range** of f is denoted $f(A) = \{f(a) : a \in A\} \subseteq B$
   (b) A function is **injective (or one-to-one)** if $\forall x, y \in A$ with $x \neq y$, it must be that $f(x) \neq f(y)$. Equivalently, if $f(x) = f(y)$, then $x = y$.
   (c) A function is **surjective (or onto)** if $f(A) = B$. That is, $\forall b \in B, \exists a \in A, f(a) = b$
   (d) A function is **bijective** if it is both injective and surjective. That is, for all $b \in B$, there is a *unique* $a \in A$ with $f(a) = b$.
   (e) Let $X \subseteq \mathcal{U}$ be a subset of some universal set. The **indicator function** for X is the function $1_X : \mathcal{U} \to \{0, 1\}$ such that $1_X(x) = 1$ if $x \in X$ and $1_X(x) = 0$ if $x \notin X$

4. **Product Rule:** Suppose there are m_1 possible outcomes for event A_1, then m_2 possible outcomes for event A_2,..., m_n possible outcomes for event A_n. Then there are $m_1 \cdot m_2 \cdot m_3 \cdots m_n = \prod_{i=1}^{n} m_i$ possible outcomes overall.

5. **Number of ways to order n distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

6. **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):

$$P(n,k) = \frac{n!}{(n-k)!}$$

Exercises

Several exercises below deal with a "standard" 52-card deck, such as is used in the games of bridge and poker. This deck consists of 52 cards divided into 4 suits of 13 cards each. The 4 suits are (black) spades ♠, (red) hearts ♡, (black) clubs ♣, and (red) diamonds ♢. The 13 cards ("ranks") of each suit are 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A.

1. **(Not covered yet)** How many ways are there to select 5 cards from a standard deck of 52 cards, where the 5 cards contain cards from at most two suits, if:

   (a) order does not matter

   Case 1: all from the same suit – choose 1 of 4 suits, and 5 cards from that suit

$$\binom{4}{1}\binom{13}{5}$$

   Case 2: from two suits — choose 2 of 4 suits: 1 from the first and 4 from the second, 2 from the first and 3 from the second, etc.)

$$\binom{4}{2} \cdot \left[\binom{13}{1}\binom{13}{4} + \binom{13}{2}\binom{13}{3} + \binom{13}{3}\binom{13}{2} + \binom{13}{4}\binom{13}{1}\right]$$

   Our total is $\boxed{\binom{4}{1}\binom{13}{5} + \binom{4}{2} \cdot \left[\binom{13}{1}\binom{13}{4} + \binom{13}{2}\binom{13}{3} + \binom{13}{3}\binom{13}{2} + \binom{13}{4}\binom{13}{1}\right]}$

   Let's talk about an **incorrect** solution:
   Step 1: First choose the two suits from which the cards will come: $\binom{4}{2}$ possibilities
   Step 2: Then choose the 5 cards from among the 26 possible cards of those suits: $\binom{26}{5}$
   Thus, the total number is ways is $\binom{4}{2}\binom{26}{5}$

Why this is wrong:

The problem is that this method overcounts some choices. In particular, a choice consisting of cards that are entirely from one suit, say hearts, will be counted 3 times:

Once when

- the two suits selected are Hearts/Spades, once when
- once when the two suits selected are Hearts/Diamonds, and once when
- the two suits selected are Hearts/Clubs

Applying "The Sleuth Principle", given an outcome selected according to some application of the product rule, we need to be able to reconstruct exactly what choice was made at each step, or else we have made a mistake. When we see an outcome consisting of all hearts, we cannot reconstruct the choice made in the first step – it could have been any of the 3 possibilities mentioned above.

To correct this, one can subtract off the overcounted stuff which is

$$2\binom{4}{1}\binom{13}{5}$$

(b) order matters

Just 5! times the previous answer, since we can permute the 5 distinct cards that many ways. $\boxed{5! \cdot \left(\binom{4}{1}\binom{13}{5} + \binom{4}{2} \cdot \left[\binom{13}{1}\binom{13}{4} + \binom{13}{2}\binom{13}{3} + \binom{13}{3}\binom{13}{2} + \binom{13}{4}\binom{13}{1}\right]\right)}$

2. **(Not covered yet)** Consider a set of 25 people that form a social network. (The structure of the social network is determined by which pairs of people in the group are friends.) How many possibilities are there for the structure of this social network?

   There are $\binom{25}{2}$ possible undirected edges representing friendships, and each is either there or not, so the number is $\boxed{2^{\binom{25}{2}}}$

3. Suppose we have 3 diamonds and 3 hearts from a standard deck. How many ways are there to arrange the cards if they have to alternate suit?

   Method 1: 6 possible cards for the first location, then 3 because you can't choose the same suit. Then 2 for the third location, because the suit is determined by the first location and there are only 2 cards left in that suit. Similarly 2 for the fourth location. Then 1 choice for each of the fifth and sixth locations. $\boxed{6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1}$

Method 2: Find the arrangements individually for each of the suits: 3! for each suit. They have to be alternating, but there are 2 choices for which suit comes first, and then the order will be determined. $\boxed{2 \cdot (3!)^2}$

Check that the answers are equivalent.

4. How many ways are there to choose three initials that have two being the same or all three being the same?

   Complementary counting. Count the total 26^3 and subtract the number with all distinct initials $26 \cdot 25 \cdot 24 = P(26, 3)$ to get $\boxed{26^3 - P(26, 3)}$.

5. A license plate has the form AXYZBCD, where A, B, C, and D are digits and X, Y, and Z are upper case letters. What is the number of different license plates that can be created?

   $\boxed{10^4 \cdot 26^3 = 175,760,000}$