CSE 312: Foundations of Computing II
Additional Exercises #1: Sets, Functions, Permutations (solutions)

Note: These exercises are strictly for your own benefit, in case you need extra practice for exams and homework.

Several exercises below deal with a "standard" 52-card deck, such as is used in the games of bridge and poker. This deck consists of 52 cards divided into 4 suits of 13 cards each. The 4 suits are (black) spades ♠, (red) hearts ♡, (black) clubs ♣, and (red) diamonds ♢. The 13 cards ("ranks") of each suit are 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A.

1. A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert combinations are there for the week?

   Start from Thursday and work forward and backward in the week: $4{\cdot}4{\cdot}4{\cdot}4{\cdot}1{\cdot}4{\cdot}4 = \boxed{4^6 = 4096}$

2. A store has 4 books, 14 movies, 6 toys, and 5 posters. In how many ways can a customer buy exactly 1 item from each of exactly 3 categories?

   $4 \cdot 14 \cdot 6 + 4 \cdot 14 \cdot 5 + 4 \cdot 6 \cdot 5 + 14 \cdot 6 \cdot 5 = \boxed{1156}$

   Ok, this was a bit hacky: we go through cases depending on which item was excluded.

3. In how many different ways can you arrange seven people around a circular table?

   $7!/7 = 6! = 720$. In general for n objects arranged in a circle, the answer is $n!/n = (n-1)!$: if you imagine the $n!$ permutations of the objects in a linear sequence, this counts each of the circular arrangements n times, because there are n different places you can "cut" the circle to get a different linear arrangement.

4. Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

   $\boxed{(2 \cdot 7 + 2 \cdot 6)6! = 18720}$ There are two cases:

   Case 1: You are next to your friend. Then there are there are 7 different slots. Then, there are 7 sets of positions you and your friend can occupy (positions 1/2, 2/3, ..., 7/8), and for each set of positions, there are 2 ways to arrange you and your friend, So there are $7 \cdot 2$ ways to pick positions for you and your friend.

Case 2: There is exactly 1 person between you and your friend. Then, there are 6 sets of positions you and your friend can occupy (positions 1/3, 2/4, ... , 6/8), and for each set of positions, there are 2 ways to arrange you and your friend. So there are $6 \cdot 2$ ways to pick positions for you and your friend.

Note that in both cases, there are then 6! ways to arrange the remaining people, so we multiply both cases by 6!.

5. Your CSE 312 teaching staff (Professor Rao and 6 TAs) lines up for a picture. How many possible arrangements are there with Professor Rao not at either end of the line?

   $\boxed{7! - 2 \cdot 6! = 3600}$

   Complementary counting. There are 7! total arrangements, but $2 \cdot 6!$ have Professor Rao at an end of the line (which is not allowed), so remove those from the count.

6. How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

   $\boxed{8! - 2 \cdot 7! + 6! = 30960}$

   This is a more complex form of complementary counting. There are 8! total arrangements. We need to subtract the 7! arrangements where A is at the beginning, and the 7! arrangements where H is at the end. But in doing this, we accidentally subtracted the arrangements where **both are true**, twice. To compensate, we need to add those back in once, which is 6!.

7. There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

   $\boxed{(10!/5!)35!}$

   Seat the students who must sit in the front row first. There are $10 * 9 * 8 * 7 * 6$ (or $10!/5!$) ways to assign seats to those students. Then there are 35 students and 35 seats left, so there are 35! ways to assign seats to the other students.

8. Permutations of objects, some of which are indistinguishable.

   (a) How many permutations are there of the letters in DAWGY? $\boxed{5! = 120}$

   (b) How many permutations are there of the letters in DOGGY? $\boxed{5!/2! = 60}$

   (c) How many permutations are there of the letters in GODOGGY? $\boxed{7!/(3!2!1!1!) = 420}$

9. A bridge hand consists of 13 cards dealt from a shuffled standard deck of 52 cards. Given a bridge hand consisting of 5 spades, 2 hearts, 3 diamonds, and 3 clubs, in how many ways can the hand be arranged so that the cards of each suit are together ...

   (a) ...but not necessarily sorted by rank within each suit? $\boxed{4!5!2!3!3! = 207360}$

   - 4! ways to order all the suits
   - 5! ways to order the spades
   - 2! ways to order the hearts
   - 3! ways to order the diamonds
   - 3! ways to order the clubs

   (b) ...and each suit is sorted in ascending rank order? $\boxed{4! = 24}$

   (c) ...and each suit is sorted in ascending rank order and the suits are arranged so that the suit colors alternate? $\boxed{4 \cdot 2 \cdot 1 \cdot 1 = 8}$ (4 options for what suit is first, 2 options for the next suit because it has to be of the other color, then one option each for the remaining two suits)

10. Suppose two cards are drawn in order from a bridge deck. In how many ways can the first card be a diamond and the second card a jack?

    $\boxed{13 \cdot 4 - 1 = 51}$

11. Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

    If Peter and Pauline go to the same store, there are 4 stores it could be. For each such choice, there are 3 choices of store for each of the 3 offspring, so 3^3 choices for all the offspring. If Peter and Pauline go to different stores, there are $4 \cdot 3 = 12$ pairs of stores they could go to. For each such choice, there are 2 choices of store for each of the 3 offspring, so 2^3 choices for all the offspring. Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3 = 204}$.

12. You have a triangular prism with top and bottom both being congruent equilateral triangles and the three sides being congruent rectangles. If you pick 5 out of 7 different colors, one to paint each of the 5 faces, how many differently painted triangular prisms can you get? Just

rotating the prism does not constitute a different color scheme.

$$\frac{7!}{2! \cdot 2 \cdot 3} = 420$$

There are $2 \cdot 3$ rotations of the prism that leave the 5 faces in their original positions. That means that $P(7, 5)$ counts each color scheme 6 times.

13. How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if . . .

    (a) . . . the seats are assigned arbitrarily?

    $\boxed{10!}$

    (b) . . . all couples are to get adjacent seats?

    $\boxed{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 2^5 \cdot 5!}$: there are 5! permutations of the 5 couples, and then 2 permutations within each of the 5 couples.

    (c) . . . the seats are assigned arbitrarily, except that one couple insists on not sitting in adjacent seats?

    There are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a ninth unit added to the other 8 individuals, and then there are 2 permutations of that couple's seats. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.