Lecture 8: Working with Bayes' Rule

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We discuss several examples of how one can use Bayes' rule.

There are many obvious identities that probabilities satisfy:

Fact 1 (Bayes' Rule)**.** *If A, B are events, then*

$$p(A|B) = \frac{p(B|A) \cdot p(A)}{p(B)}.$$

Fact 2 (Chain Rule)**.** *If $A_1, A_2, \dots, A_n$ are events, then*

$$p(A_1 \cap A_2 \cap \dots A_n)$$
$$= p(A_1) \cdot p(A_2|A_1) \cdot p(A_3|A_1 \cap A_2) \cdot \dots \cdot p(A_n|A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

Fact 3 (Law of Total Probability)**.** *If $A_1, A_2, \dots, A_n$ are disjoint events that form a partition of the whole sample space, and B is another event, then*

$$p(B) = p(A_1 \cap B) + p(A_2 \cap B) + \dots + p(A_n \cap B).$$

Exampe: Using Bayes' Rule

Suppose an urn either contains 3 red balls and 3 blue balls with probability 3/4, or 6 red balls with probability 1/4. You draw 3 balls at random and come up with 3 red balls. What are the remaining balls in the urn?

Let M denote the event that the balls are mixed. Let D denote the event that 3 red balls were drawn. We have

$$p(D|M) = \binom{3}{3} / \binom{6}{3} = 1/20.$$

and

$$p(M|D) = \frac{p(D|M)p(M)}{p(D)}.$$

We can calculate $p(D) = p(D|M)p(M) + p(D|M^c)p(M^c) = \frac{1}{20} \cdot 3/4 + 1 \cdot 1/4 = 23/80$. Then we get

$$p(M|D) = \frac{p(D|M)p(M)}{p(D)} = \frac{(1/20)(3/4)}{23/80} = 3/23.$$

$p(M)$ is often called the *prior*. $p(M|D)$ is called the *posterior*.

We start by discussing a simple application of Bayes' rule.

Example: Radar

Suppose the airforce designs a new radar system. If an aircraft is present in the range of the radar system, then the aircraft is detected with probability 0.99. If the aircraft is not present, then the radar reports that an aircraft is present with probability 0.1. Suppose the probability than an aircraft is present is 0.05. What is the probability that the system gives a false alarm, meaning that an aircraft is not presented but is detected? What is the probability that an aircraft is present and detected? What is the probability that an aircraft is present given that the radar reports an aircraft?

The first thing to do is to model all the events we care about with events. Let A be the event that an aircraft is present, and R be the event that the radar detects an aircraft. Then we have $p(R|A) = 0.99$ and $p(R|A^c) = 0.1$. Finally, we know that $p(A) = 0.05$. The probability of a false alarm is $p(A^c \cap R)$. We have:

$$p(A^c \cap R) = p(A^c) \cdot p(R|A^c) = (1 - 0.05) \cdot 0.1 = 0.095.$$

Similarly, we have

$$p(A \cap R) = p(A) \cdot p(R|A) = 0.05 \cdot 0.99 = 0.0495.$$

The probability that there is an aircraft given that the radar reports one can be calculated using Bayes' rule:

$$p(A|R) = \frac{p(R|A)p(A)}{p(R)}.$$

We see that we know all of the quantities on the right hand side except $p(R)$. However, we have

$$\begin{aligned} p(R) &= p(R \cap A) + p(R \cap A^c) \\ &= 0.095 + 0.0495 = 0.1445. \end{aligned}$$

So, we get

$$p(A|R) = \frac{p(R|A)p(A)}{p(R)} = \frac{0.0495}{0.1445} = 0.34256,$$

which is not as high as you might expect. The point is that because the probability of an aircraft being present is so low, the probability of a false alarm from the radar better be extremely low for the radar to be effective.

Example: Gamblers Ruin

Suppose a gambler has $0 < i < N$ dollars. In each step, with probability $1/2$ the gambler makes a dollar, and with probability $1/2$ the gambler loses a dollar. If the gambler ever hits 0 dollars she loses. If she hits N dollars, she wins. What is the probability she wins?

Let E_i denote the event that the probability that the gambler wins starting with i dollars, and let $p_i = p(E_i)$. Then we have $p_0 = 0, p_N = 1$, and for $0 < i < N$,

$$p_i = p(E_i) = p(E_{i+1}) \cdot \frac{1}{2} + p(E_{i-1}) \cdot \frac{1}{2} = \frac{1}{2}(p_{i-1} + p_{i+1}).$$

Rearranging, we get

$$p_{i+1} = 2p_i - p_{i-1}.$$

So $p_2 = 2 \cdot p_1 - p_0 = 2p_1$, $p_3 = 2p_2 - p_1 = 3p_1$, and in general

$$p_{i+1} = 2p_i - p_{i-1} = 2ip_1 - (i-1)p_1 = (i+1)p_1.$$

So we have $1 = p_N = Np_1$. This gives $p_1 = 1/N$, and $p_i = i/N$ for all i.

Suppose the gambler is really addicted to gambling, and suppose in each step he wins a dollar with probabilty $1/2$, and loses a dollar with probability $1/2$. However, this time, the gambler doesn't stop when he has N dollars. He continues to gamble forever. What is the probability that he loses? Is there a chance that he can keep winning forever?

Let p_i denote the probability that the gambler wins. Then we have:

$$p_i = \frac{p_{i+1} + p_{i-1}}{2},$$

so as before, we get

$$p_{i+1} = 2p_i - p_{i-1}.$$

One possible solution to these equations is that $p_i = 0$ for all i—the gambler always loses, no matter how money he starts with. Indeed, this is the only solution. Certainly if $p_1 = 0$, then $p_2 = 2p_1 - p_0 = 0$, and in this way $p_i = 0$ for all i. On the other hand, if $p_1 > 0$, then exactly as before, we have $p_i = ip_1$, which implies for large enough i, $p_i > 1$, which is impossible. This proves that $p_i = 0$ is the only solution.

What happens if the gambler never quits, but wins each bet with probability 0.99? Even in this case you can show that he eventually loses, though this requires some additional work.

The proof that I have in mind requires the concept of *variance*, which we shall discuss in a future lecture.