Lecture 3: Combinations

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We continue our discussion of counting. Here we introduce binomial coefficients and discuss several identities and estimates involving them.

Combinations

So far, we have been counting the number sequences one can generate using a set. Let us now turn to counting the number of sets.

Example

How many sets of 5 English letters are there?

If we had to count the number of 5 letter words, we know that the answer is $P(26, 5) = \frac{26!}{21!}$. This is not what we want, because it counts ABCDE as different from BACDE, while the sets $\{$A,B,C,D,E$\}$ and $\{$B,A,C,D,E$\}$ are equal. However, we see that we have overcounted by exactly 5!. This is because there are 5! different ways to rearrange ABCDE, and all of these give the same set of letters. So, the correct answer is $\frac{26!}{5! \cdot 21!}$.

In general, if the number of subsets of $[n]$ of size k is exactly

$$\binom{n}{k} = P(n,k)/k! = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot 1}.$$

$\binom{n}{k}$ is verbalized *n choose k*.

Example

How many ways are there to rearrange the letters of the word AGAGGLE?

We have already seen one way to compute the answer—there are 7! ways to permute all the letters, but for each of these, $3! \cdot 2!$ of them correspond to the same arrangement of G's and A's, so we have overcounted by $3! \cdot 2!$. The final answer is thus $\frac{7!}{3!}$.

Here is a different way to count the same quantity. There are $\binom{7}{3}$ ways to place the G's. Once they have been placed, there are $\binom{4}{2}$ ways to place the A's, and once they have been placed, there are 2 ways to arrange the L and E. This gives

$$\binom{7}{3} \cdot \binom{4}{2} \cdot 2 = \frac{7! \cdot 4! \cdot 2}{3! \cdot 4! \cdot 2! \cdot 2!} = \frac{7!}{3! \cdot 2!}.$$

So, we see that the answer is the same as before, but we arrived at it differently.

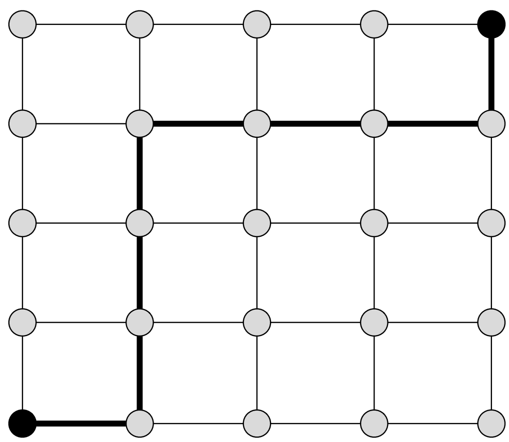
Example

How many ways are there to distribute 10 dollar notes to 3 people?

The question above is the same as the number of solutions to the equation $x_1 + x_2 + x_3 = 10$. To count this, we use a trick. Each assignment of dollars corresponds to a sequence over a binary alphabet. For example $x_1 = 4, x_2 = 5, x_3 = 1$ corresponds to $\$\$\$\$ * \$\$\$\$\$ * \$$. So, the number of ways to do the arrangement is the same as the number of ways to pick 2 $*$'s out of the $10 + 2$ positions. So the answer is $\binom{12}{2}$. In general, if we have an equation $\sum_{i=1}^{k} x_i = n$, the number of non-negative integer solutions is $\binom{n+k-1}{k-1}$.

Example

How many ways are there to walk from bottom left to the top right corners in this 5×5 grid if you are only allowed to move right or up?



To reason about this, observe that every walk on this grid corresponds to a sequence of moves that either right or up. For example, the depicted walk corresponds to $RUUURRRU$. Since we have to end up at the top-right corner, there must be exactly 4 R's and 5 U's. So, the number of such paths is equivalent to the number of ways to set 4 out of 8 locations in the string to be R. The count is then $\binom{8}{4}$.

What is the count if the grid was an $m \times n$ grid? Answer: $\binom{m+n-2}{m-1}$.

Example

How many ways are there to walk from bottom left to the top right corners in the $(n+1) \times (n+1)$ grid if you are must always stay below the diagonal? We know that the total number of ways to reach the point (n, n) is $\binom{2n}{n}$. Now we want to eliminate the ways that cross the diagonal. So it is enough count how many path do actually cross the diagonal. Look at the first time a path cross the diagonal. At this point, the path must have taken an odd number $2j + 1$ steps, with $j +$

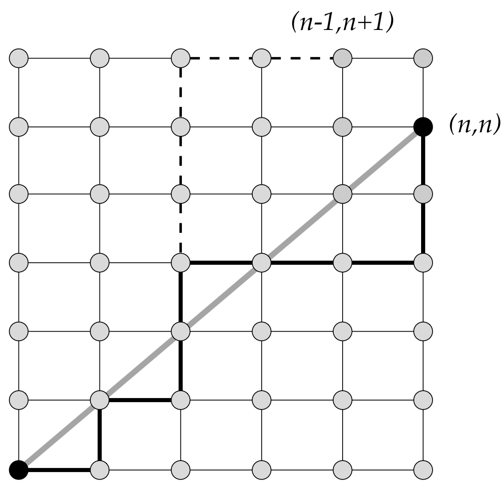


Figure 1: A path that crosses the diagonal can be reflected, so that it reaches the point

1 of them being up and j of them being right. If we now reflect the rest of the path by switching the remaining right steps to up and the remaining up steps to right, we will end up with $j + 1 + n - j = n + 1$ steps up and $j + n - (j + 1) = n - 1$ steps to the right. Moreover, this process is reversible: any path that goes to $(n - 1, n + 1)$ correspond to a path that goes to (n, n) and crosses the diagonal—just reflect the path after the first time that it crosses the diagonal.

So, the total number of paths that do not cross the diagonal is

$$\binom{2n}{n} - \binom{2n}{n-1}.$$

Later, we shall show that this number is the same as $\frac{1}{n+1} \cdot \binom{2n}{n}$.