Lecture 10: More Examples with Conditional Probability, and Using Random Variables

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We see a few more examples using conditional probability, and some examples where random variables are very useful.

Recall these basic facts:

**Fact 1 (Chain Rule).** If $A_1, A_2, \ldots, A_n$ are events, then

$$p(A_1 \cap A_2 \cap \ldots A_n) = p(A_1) \cdot p(A_2|A_1) \cdot p(A_3|A_1 \cap A_2) \cdot \ldots \cdot p(A_n|A_1 \cap A_2 \cap \ldots A_{n-1}).$$

The chain rule can make it easy to compute probabilities:

**Example: Dealing Aces**

Suppose you split a standard deck of cards randomly into 4 parts. What is the probability that each of the aces is in a different part?

Let $A_1$ be the event that the aces of hearts and spades are in different piles. Let $A_2$ be the event that the aces of hearts, spades and diamonds are in different piles, and let $A_3$ be the event that all aces are in different piles.

Then we want to calculate

$$p(A_3) = p(A_1 \cap A_2 \cap A_3) = p(A_1) \cdot p(A_2|A_1) \cdot p(A_3|A_1 \cap A_2).$$

We have

$$p(A_1) = \frac{39}{51},$$

because for every placement of the ace of hearts, there are 51 locations where the ace of spades can go, and 39 of them would be in a different part. Similarly, we have

$$p(A_2|A_1) = \frac{26}{50},$$

and

$$p(A_3|A_1 \cap A_2) = \frac{13}{49}.$$

So, we conclude that

$$p(A_3) = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} \approx 0.105.$$
Example: GPA

Barack will get an A in physics with probability 3/4, and will get an A in chemistry with probability 3/5. He flips a coin and takes one of the courses based on his coin flip. What is the probability he gets an A?

If \( A \) denote the event that he gets an A, \( P \) denotes the event that he takes physics and \( C \) denotes the event that he takes Chemistry, we see that \( P, C \) are a partition of the probability space, so

\[
p(A) = p(A|P) \cdot p(P) + p(A|C) \cdot p(P) = \frac{3}{8} + \frac{3}{10} = \frac{27}{40}.
\]

Fact 2 (Law of Total Probability). If \( A_1, A_2, \ldots, A_n \) are disjoint events that form a partition of the whole sample space, and \( B \) is another event, then

\[
p(B) = p(A_1 \cap B) + p(A_2 \cap B) + \ldots + p(A_n \cap B).
\]

One consequence of the Law of Total Probability is that whenever you have a random variable \( X \) and an event \( E \), we have

\[
p(E) = \sum_x p(X = x) \cdot p(E|X = x).
\]

Example: Even Number of Heads

Suppose you toss \( n \) coins. What is the probability that the number of heads is even?

We already calculated this the hard way. Let us use an appropriate random variable to calculate this the easy way.

Let \( X \) denote the outcome of tossing the first \( n - 1 \) coins. Let \( E \) denote the event that the number of heads is even. Then:

\[
p(E) = \sum_{x \in \{H,T\}^{n-1}} p(X = x) \cdot p(E|X = x).
\]

Now observe that no matter what \( x \) is, \( p(E|X = x) = 1/2 \). This is because after you fix the first \( n - 1 \) coin-tosses, the last coin toss is still uniformly random. If \( x \) has an even number of heads, then the total number of tosses is even when the last coin toss is a tail and so \( p(E|X = x) = 1/2 \). If \( x \) has an odd number of heads, the opposite is true, and still \( p(E|X = x) = 1/2 \). So, we get:

\[
p(E) = \sum_{x \in \{H,T\}^{n-1}} p(X = x) \cdot p(E|X = x)
= (1/2) \cdot \sum_{x \in \{H,T\}^{n-1}} p(X = x) = 1/2.
\]