Lecture 10: More Examples with Conditional Probability, and Using Random Variables

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We see a few more examples using conditional probability, and some examples where random variables are very useful.

Recall these basic facts:

Fact 1 (Chain Rule). *If $A_1, A_2, \ldots, A_n$ are events, then*

$$p(A_1 \cap A_2 \cap \ldots A_n)$$
$$= p(A_1) \cdot p(A_2|A_1) \cdot p(A_3|A_1 \cap A_2) \cdot \ldots \cdot p(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

The chain rule can make it easy to compute probabilities:

Example: Dealing Aces

Suppose you split a standard deck of cards randomly into 4 parts. What is the probability that each of the aces is in a different part?

You could also calculate this by explicitly counting: $\frac{4! \cdot \binom{48}{12}\binom{36}{12}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}}$. However, using conditional probability is much cleaner!

Let A_1 be the event that the aces of hearts and spades are in different piles. Let A_2 be the event that the aces of hearts, spades and diamonds are in different piles, and let A_3 be the event that all aces are in different piles.

Then we want to calculate

$$p(A_3) = p(A_1 \cap A_2 \cap A_3) = p(A_1) \cdot p(A_2|A_1) \cdot p(A_3|A_1 \cap A_2).$$

We have

$$p(A_1) = \frac{39}{51},$$

because for every placement of the ace of hearts, there are 51 locations where the ace of spades can go, and 39 of them would be in a different part. Similarly, we have

$$p(A_2|A_1) = \frac{26}{50},$$

and

$$p(A_3|A_1 \cap A_2) = \frac{13}{49}.$$

So, we conclude that

$$p(A_3) = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} \approx 0.105.$$

Example: GPA

Barack will get an A in physics with probability 3/4, and will get an A in chemistry with probability 3/5. He flips a coin and takes one of the courses based on his coin flip. What is the probability he gets an A?

If A denote the event that he gets an A, P denotes the event that he takes physics and C denotes the event that he takes Chemistry, we see that P, C are a partition of the probability space, so

$$p(A) = p(A|P) \cdot p(P) + p(A|C) \cdot p(P) = 3/8 + 3/10 = 27/40.$$

Fact 2 (Law of Total Probability)**.** *If $A_1, A_2, \ldots, A_n$ are disjoint events that form a partition of the whole sample space, and B is another event, then*

$$p(B) = p(A_1 \cap B) + p(A_2 \cap B) + \ldots + p(A_n \cap B).$$

One consequence of the Law of Total Probability is that whenever you have a random variable X and an event E, we have

$$p(E) = \sum_x p(X = x) \cdot p(E|X = x).$$

Example: Even Number of Heads

Suppose you toss n coins. What is the probability that the number of heads is even?

We already calculated this the *hard* way. Let us use an appropriate random variable to calculate this the easy way.

See notes for Lecture 6 on January 17.

Let X denote the outcome of tossing the first $n-1$ coins. Let E denote the event that the number of heads is even. Then:

$$p(E) = \sum_{x \in \{H,T\}^{n-1}} p(X = x) \cdot p(E|X = x).$$

Now observe that no matter what x is, $p(E|X = x) = 1/2$. This is because after you fix the first $n-1$ coin-tosses, the last coin toss is still uniformly random. If x has an even number of heads, then the total number of tosses is even when the last coin toss is a tail and so $p(E|X = x) = 1/2$. If x has an odd number of heads, the opposite is true, and still $p(E|X = x) = 1/2$. So, we get:

$$\begin{aligned} p(E) &= \sum_{x \in \{H,T\}^{n-1}} p(X = x) \cdot p(E|X = x) \\ &= (1/2) \cdot \sum_{x \in \{H,T\}^{n-1}} p(X = x) = 1/2. \end{aligned}$$