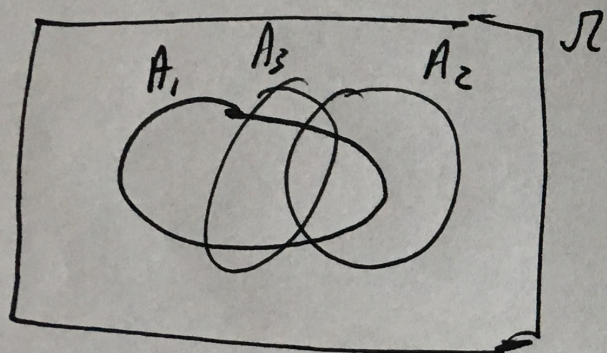


Probability Space

Ω, P

- For all $x \in \Omega$, $p(x) \geq 0$

- $\sum_{x \in \Omega} p(x) = 1$.



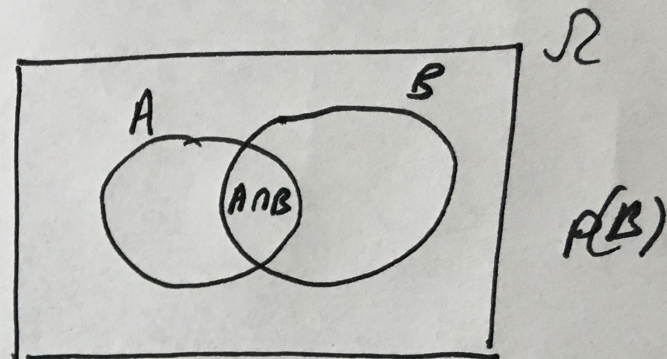
$A_1, \dots, A_n$: n events

$$P(A_1 \cup \dots \cup A_n)$$

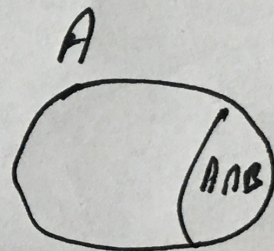
$$= P(A_1) + P(A_2) + \dots + P(A_n)$$

$$- P(A_1 \cap A_2) - \dots - P(A_{n-1} \cap A_n) \\ + P(A_1 \cap A_2 \cap A_3) + \dots$$

Conditional Probability



$\downarrow$ A happened



What is prob. B also happened?

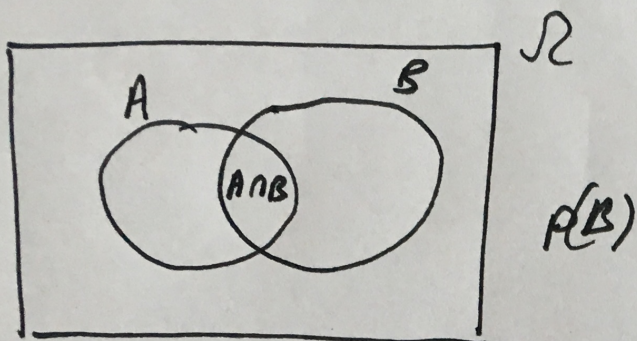
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Intuition: new prob. space

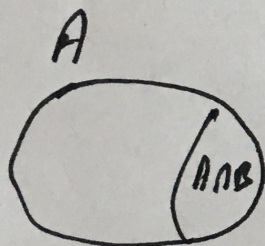
sample set: A

$$\text{distribution: } q(x) = \frac{P(x)}{P(A)}$$

Conditional Probability



↓ A happened



What is prob. B also happened?

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Intuition: new prob. space

sample set: A

distribution: $q(x) = \frac{P(x)}{P(A)}$

~~For~~ For all $x \in A$,

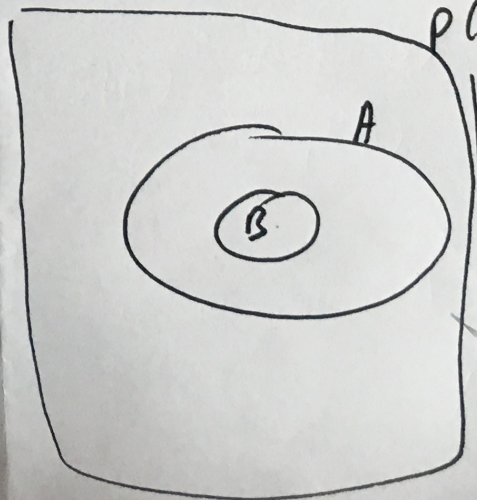
$$q(x) \geq 0?$$

$$\frac{P(x)}{P(A)} \geq 0$$

$$\begin{aligned} \sum_{x \in A} q(x) &= \sum_{x \in A} \frac{P(x)}{P(A)} \\ &= \frac{1}{P(A)} \cdot \sum_{x \in A} P(x) \end{aligned}$$

$$= \frac{P(A)}{P(A)} = 1$$

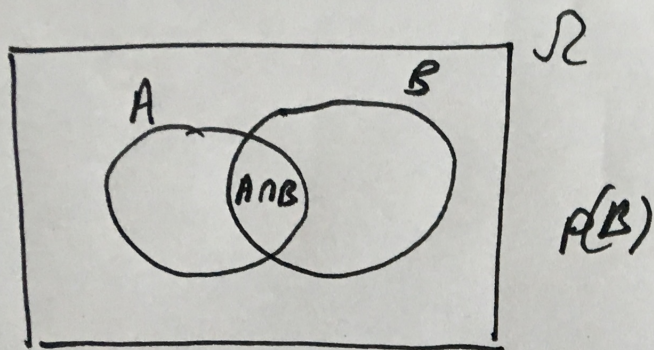
$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \Bigg| \quad P(A|B) = \frac{P(A \cap B)}{P(B)}$$



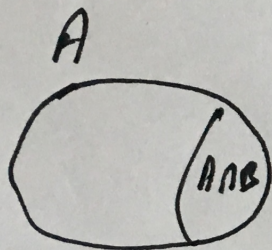
$$P(A|B) = 1$$

$$P(B|A) < 1$$

Conditional Probability



↓ A happened



What is prob. B also happened?

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Intuition: new prob. space

sample set: A

distribution: $q(x) = \frac{P(x)}{P(A)}$

Bayes Rule

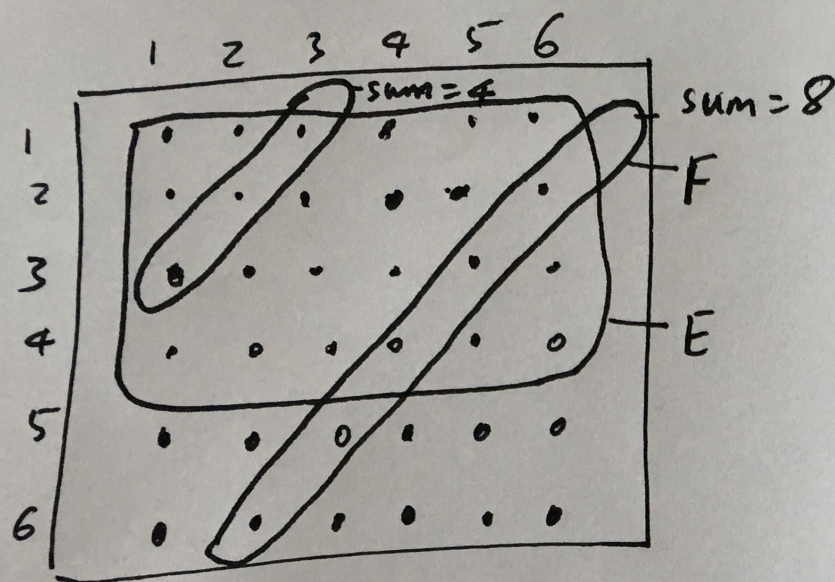
$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow \cancel{P(A \cap B)} = \cancel{P(A)} \cdot P(B|A)$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$P(A) \cdot P(B|A) = P(A \cap B) = P(B) \cdot P(A|B)$$

$$\Rightarrow \boxed{P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}}$$

Roll Two Dice



F : event that sum = 9

$$P(F) = \frac{5}{36} = 0.1388$$

E : event that first die ≤ 4

$$P(E) = \frac{24}{36} = \frac{2}{3} = 0.666..$$

$$P(E^c) = \frac{1}{3} = 0.333$$

$$P(E \cap F) = \frac{3}{36} = \frac{1}{12}$$

What is prob sum = 9 given first die ≤ 4 ?

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{1/12}{2/3}$$

$$= \frac{3}{2 \cdot 12} = \frac{1}{8} = 0.125$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/12}{5/36}$$

$$= \frac{36}{5 \cdot 12} = \frac{3}{5} = 0.6$$

$$P(E^c|F) = 0.4$$

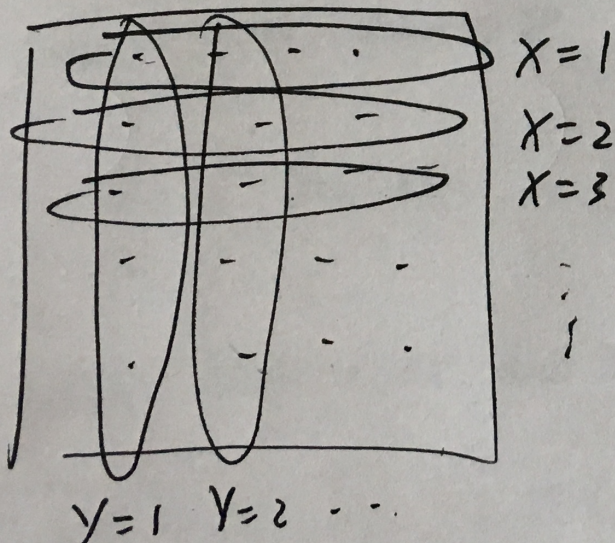
Random Variables

$$X: \Omega \rightarrow S$$

2 Dice

X : value of first die

Y : value of second die



$p(X, Y) \leftarrow$ joint distribution

$p(X) \leftarrow$ marginal distribution

$$Z = X + Y$$

E : event that $X \leq 4$

F : event that $X + Y = 8$

Independent

For all x, y

$$p(X=x \text{ and } Y=y) = p(X=x) \cdot p(Y=y)$$

$$p(X=x) \cdot p(Y=y | X=x)$$

$$\Leftrightarrow p(Y=y | X=x) = p(Y=y)$$