Pigeonhole Principle

$n+1$ pigeons $\rightarrow$ n holes

2 pigeons in same hole.

Intersecting Families

$S_1, S_2, \dots, S_m \subseteq [n]$

are intersecting $\forall\, i,j \quad S_i \cap S_j \neq \phi$

$[4]$

$$\underbrace{\{1\}, \{1,2\}, \{1,2,3\}, \{1,3\}, \{1,4\}, \{1,2,3,4\}, \{1,3,4\}, \{1,2,4\}}_{8 \text{ sets}}$$

$[4]$

$$\underbrace{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}}_{5 \text{ sets}}$$

$[n]$

$\mathcal{F}$: all subsets containing 1.

$$|\mathcal{F}| = 2^{n-1} = 2^n/2$$

<u>Claim:</u> If $\mathcal{F}$ is intersecting then $|\mathcal{F}| \leq \dfrac{2^n}{2}$.

<u>Pf:</u> Suppose $|\mathcal{F}| \geq \dfrac{2^n}{2} + 1$.

$\mathcal{F}$: set of pigeons.

$\phi, [n]$ $\{1\}, [n]-\{1\}$ $\dots$ A, A^C] $\dfrac{2^n}{2}$ holes

Pig. princ.: 2 sets in $\mathcal{F}$ belong to same hole! $\Rightarrow \Leftarrow$

Probability

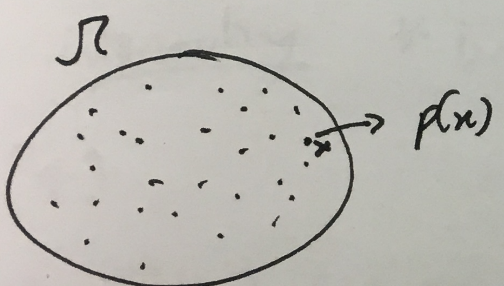
(Discrete) Probability space

sample set : $\Omega \leftarrow$ outcomes

distribution : $p : \Omega \to \mathbb{R}$

s.t. · For every $x \in \Omega$, $p(x) \geq 0$.

· $\sum_{x \in \Omega} p(x) = 1$.



1 coin toss

$\Omega = \{H, T\}$

$p(H) = p(T) = 1/2$.

2 coin tosses

$\Omega = \{HH, HT, TH, TT\}$

$p(HT) = 1/4$.

Event

$E \subseteq \Omega$

E : two coins give same value

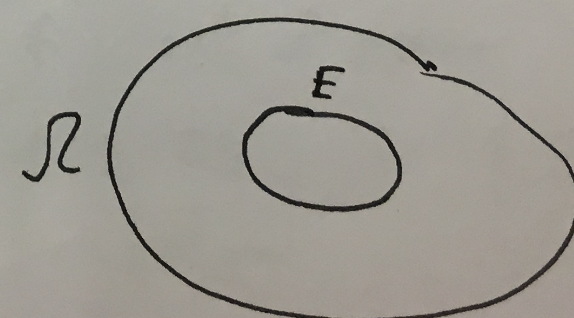
$= \{HH, TT\}$

$$p(E) = \sum_{x \in E} p(x)$$

Uniform distribution : For all $x, y \in \Omega$

$p(x) = p(y)$

$$p(E) = \sum_{x \in E} p(x) = \sum_{x \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$



n coin tosses

What is the prob. that you get even # heads?

E : set of tosses with even # heads

Ω : n coin tosses

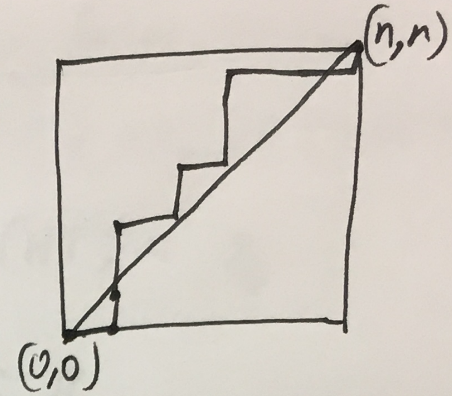
$$|\Omega| = 2^n$$

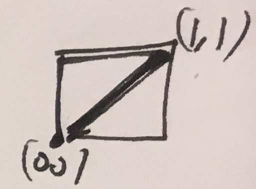
$$|E| = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

$$\frac{|E|}{|\Omega|} = \frac{\binom{n}{0} + \binom{n}{2} + \cdots}{2^n}$$

$$= \frac{\binom{n}{0} + \binom{n}{2} + \cdots}{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots}$$

$$= \frac{\binom{n}{0} + \binom{n}{2} + \cdots}{2\left(\binom{n}{0} + \binom{n}{2} + \cdots\right)} = \frac{1}{2}$$





If you pick a path from (0,0) to (n,n) uniformly at random, what is prob. you cross diagonal?

$$|\Omega| = \binom{2n}{n}$$

E : set of paths crossing diagonal

$$|E| = \frac{n}{n+1} \cdot \binom{2n}{n}$$

$$p(E) = \frac{|E|}{|\Omega|} = \frac{\left(\frac{n}{n+1}\right) \cdot \binom{2n}{n}}{\binom{2n}{n}}$$

$$= \frac{n}{n+1}$$