Inclusion - Exclusion

$A_1, \dots, A_n \quad A_I = \bigcap_{i \in I} A_i$

$I \subseteq [n]$

<u>FACT</u>: $\left| \bigcup_{i \in [n]} A_i \right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \cdot |A_I|$

$//$

$$\overbrace{|A_1| + \cdots + |A_n|}^{\binom{n}{1}} - \left(|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|\right) \leftarrow \binom{n}{2}$$

$$+ \left(|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|\right) \leftarrow \binom{n}{3}$$

$$- \quad \vdots$$

Number of Derangement

$1, 2, \dots, n \qquad n, n-1, \dots, 1 \times$

\# ways to rearrange $1, \dots, n$ st i is <u>not</u> in i^{th} position?

A_i = # ways where i is in i^{th} position.

$|A_i| = n-1!$

$|A_1 \cap A_2| = n-2!$

$I \subseteq [n] \quad |A_I| = \left| \bigcap_{i \in I} A_i \right| = (n - |I|)!$

$$n! - \left| \bigcup_{i \in [n]} A_i \right|$$

$$= n! + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I| \cancel{+1}} |A_I|$$

$$= n! + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} (n - |I|)!$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! \qquad \left| \frac{n!}{i!(n-i)!} \right. // \binom{n}{i}$$

$$= \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} = n! \cdot \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cancel{+} - \cdots$$

$$\rightarrow \quad n! \cdot \sum_{i=0}^{n} \frac{(-1)^i}{i!} = n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots \cancel{+} + \frac{(-1)^n}{n!}\right)$$

$$\approx \frac{n!}{e}$$

Euler's Totient Function

$N = p \cdot q$

$\gcd(3p, N) = p$

How many #'s from $1, 2, \ldots, N$ are relatively prime to N?

$N = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ ← prime factorization

a, b relatively prime means $\gcd(a,b) = 1$

Example

	Rel prime	Not-rel prime
20	11, 9, 13, 3	10, 4, 8, 15

A_i = set of numbers from $1, 2, \ldots, N$ divisible by p_i

$|A_i| =$

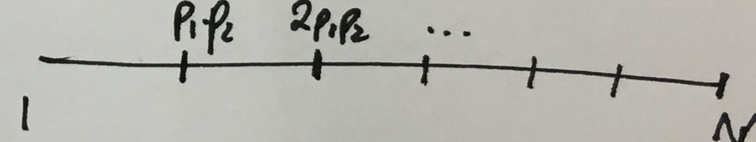
$N = 10 = p_1 p_2$, with $p_1 = 2$, $p_2 = 5$

$A_1 = \cdots$ divisibly by 2

$|A_1| = 10/2$

$A_2 = \cdots$ divisibly by 5

$|A_2| = 10/5 = 2$



$|A_1| = \dfrac{N}{p_1}$

$|A_1 \cap A_2| = \dfrac{N}{p_1 \cdot p_2}$

$|A_I| = \dfrac{N}{\prod_{i \in I} p_i}$

relatively prime

$$N - \left|\bigcup_i A_i\right|$$

$$= N - \sum_{\emptyset \neq I \subseteq [t]} (-1)^{|I|+1} |A_I|$$

$$= \sum_{I \subseteq [t]} (-1)^{|I|} \frac{N}{\prod_{i \in I} p_i}$$

$$= N \prod_{i=1}^{t} \left(1 - 1/p_i\right)$$

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$$

$$= 1 - \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_1 p_2}$$