CSE 312

Maximum Likelihood Estimation

Parameter estimation

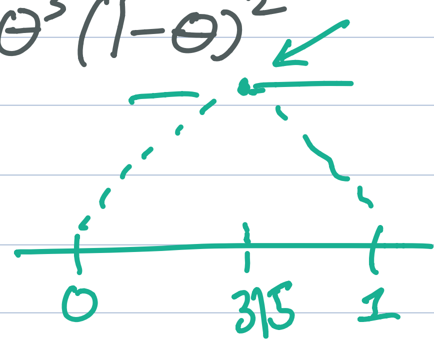
Model: Queries $\sim$ Poisson(λ)

Height of penguins

$\sim N(\mu, \sigma^2)$

Suppose I flip a coin 5 times and it comes up heads 3 times.

$$L(3 \mid \theta) = \binom{5}{3} \theta^3 (1-\theta)^2$$



$$\frac{d}{d\theta} L(3 \mid \theta)$$

$$= \binom{5}{3} \Big[3\theta^2 (1-\theta)^2 - 2(1-\theta)\theta^3 \Big]$$

$$= 0$$

$$3\cancel{\theta^2}(1-\theta)^2 = 2\cancel{(1-\theta)}\theta^3$$

$$3(1-\theta) = 2\theta \qquad 5\theta = 3$$

$$\theta = 3/5$$

$$\theta^* = 3/5$$

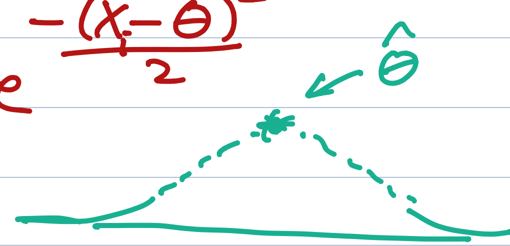
$N(\mu, \sigma^2)$

density: $\dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\underline{\sigma = 1}$ $\qquad \theta = \mu$ is the unknown parameter

$$L(x_1, x_2, \ldots, x_n \mid \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}}$$



$$\log L(x_1, x_2, \ldots, x_n \mid \theta)$$

$$= -n\log(\sqrt{2\pi}) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$\frac{d}{d\theta} \log L(x_1, \dots, x_n | \theta)$$

$$= \sum_{i=1}^{n} (x_i - \theta) = 0$$

$$\Longrightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$N(\mu, \sigma^2)$

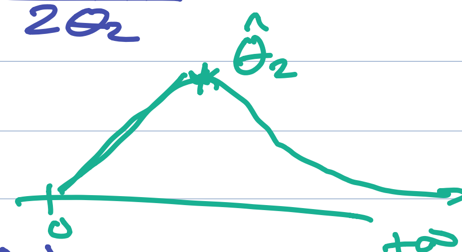
$\theta_1 \sim \mu$

$\theta_2 \sim \sigma^2$

$$\boxed{\log = \ln}$$

$$L(x_1, x_2, \dots, x_n | \theta_1, \theta_2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{\frac{-(x_i - \theta_1)^2}{2\theta_2}}$$



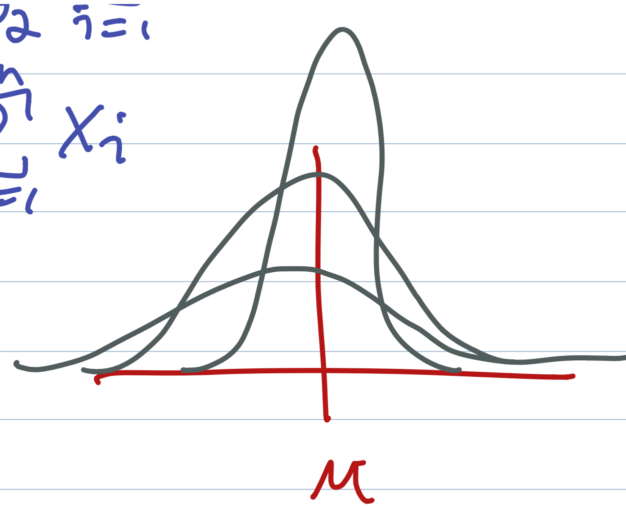
$$\frac{\partial}{\partial \theta_1} \log L(x_1, \dots, x_n | \theta_1, \theta_2)$$

$$\frac{\partial}{\partial \theta_1} \left(-\frac{n}{2} \log(2\pi\theta_2) - \frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - \theta_1)^2}{\theta_2} \right)$$

$$\frac{2\pi}{2\pi\theta_2} = \frac{1}{\theta_2}$$

$$= \frac{1}{\theta_2} \sum^{n} (x_i - \theta_1) = 0$$

$$\hat{\Theta}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$$



$$\frac{d}{d\Theta_2}(\cdots) = \frac{-n}{2\Theta_2} + \frac{1}{2}\frac{1}{\Theta_2^2}\sum_{i=1}^{n}(x_i - \Theta_1)^2$$

$$= 0$$

$$n = \frac{1}{\Theta_2}\sum_{i=1}^{n}(x_i - \Theta_1)^2$$

$$\hat{\Theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\Theta}_1)^2 \qquad \mathbb{E}[\hat{\Theta}_2] > \sigma^2$$

$$\hat{\Theta}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \mathbb{E}[\hat{\Theta}_1] = \mu$$

Consistency : As #samples $\rightarrow \infty$, estimators should converge to the true values.

"Sample variance"

$$\tilde{\Theta}_2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\Theta}_1)^2$$

$$\mathbb{E}[\tilde{\Theta}_2] = \sigma^2$$

is an unbiased estimator.