Homework 7,

Note: Problem 3 correction!

Chernoff Bound

Suppose $X_1, \dots, X_n$ are indep. r.v taking values in $\{0,1\}$ and

$$X = X_1 + X_2 + \cdots + X_n$$

if $\mathbb{E}[X] = \mu$

For all $\delta > 0$

$$p(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2+\delta}}$$

$$p(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}} \quad \leq$$

Corollary:

$$p(|X - \mu| \geq \delta\mu) \leq 2 \cdot e^{-\frac{\delta^2 \mu}{2+\delta}}$$

Fair Coin

$X_1, \dots, X_n$ n coin tosses

$X = X_1 + \cdots + X_n$ # heads.

$\mu = n/2$.

$0.6 = \frac{1.2}{2}$

$$p(X \geq 0.6\, n)$$

$$= p(X \geq (1 + \underbrace{0.2}_{\delta}) \underbrace{n/2}_{\mu})$$

$$\leq e^{-\frac{(0.2)^2}{2+0.2} \cdot n/2}$$

When $n = 200$

$$= 0.0356$$

Polling

How many people in US support Green party?

p fraction of people support Green party.

$X_1, X_2, \dots, X_n$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person supports G.P} \\ 0 & \text{o.w} \end{cases}$$

$$X = X_1 + \cdots + X_n \qquad \mathbb{E}[X] = pn$$

Our estimate X/n.

$$p(|X/n - p| \geq \theta) \leq \varepsilon$$

$$= p(|X - pn| \geq \theta n)$$

$$= p\left(|X - pn| \geq \left(\frac{\theta}{p}\right) \cdot pn\right) \qquad \delta = \theta/p$$

$$\leq 2 \cdot e^{-\frac{(\theta/p)^2 \cdot pn}{2 + \theta/p}}$$

$$= 2 \cdot e^{\frac{-\theta^2}{p2 + \theta} \cdot n}$$

$$\leq 2 \cdot e^{\frac{-\theta^2}{2+\theta} \cdot n}.$$

$$2 \cdot e^{\frac{-\theta^2}{2+\theta} \cdot n} \leq \varepsilon$$

$$\Rightarrow \quad 2/\varepsilon \leq e^{\frac{\theta^2}{2+\theta} \cdot n}$$

$$\Rightarrow \quad n \geq \left(\frac{2+\theta}{\theta^2}\right) \cdot \ln(2/\varepsilon)$$

Homework 7,

Note: Problem 3 correction!

Chernoff Bound

Suppose $X_1, \dots, X_n$ are indep. r.v taking values in $\{0,1\}$ and

$$X = X_1 + X_2 + \cdots + X_n$$

if $\mathbb{E}[X] = \mu$

For all $\delta > 0$

$$p(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2+\delta}}$$

$$p(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}} \quad \text{VI}$$

Corollary:

$$p(|X - \mu| \geq \delta\mu) \leq 2 \cdot e^{-\frac{\delta^2 \mu}{2+\delta}}$$

$$p(|X/n - p| \geq \theta) \leq \varepsilon$$

$$= p(|X - pn| \geq \theta n)$$

$$= p\left(|X - pn| \geq \left(\frac{\theta}{p}\right) \cdot pn\right) \quad \delta = \theta/p$$

$$\leq 2 \cdot e^{-\frac{(\theta/p)^2 \cdot pn}{2 + \theta/p}}$$

$$= 2 \cdot e^{\frac{-\theta^2}{p2+\theta} \cdot n}$$

$$\leq 2 \cdot e^{\frac{-\theta^2}{2+\theta} \cdot n}.$$

$$2 \cdot e^{\frac{-\theta^2}{2+\theta} \cdot n} \leq \varepsilon$$

$$\Rightarrow \quad 2/\varepsilon \leq e^{\frac{\theta^2}{2+\theta} \cdot n}$$

$$\Rightarrow \quad n \geq \left(\frac{2+\theta}{\theta^2}\right) \cdot \ln(2/\varepsilon)$$

upport

pport

i^{th} person supports G. p

$E[X] = pn$

$$p\left(\left|\frac{X}{n} - p\right| \geq \theta\right) \leq \varepsilon$$

$$= p(|X - pn| \geq \theta n)$$

$$= p\left(|X - pn| \geq \left(\frac{\theta}{p}\right) \cdot pn\right) \qquad \delta = \theta/p$$

$$\leq 2 \cdot e^{-\frac{(\theta/p)^2 \cdot pn}{2 + \theta/p}}$$

$$= 2 \cdot e^{\frac{-\theta^2}{p2+\theta} \cdot n}$$

$$\leq 2 \cdot e^{\frac{-\theta^2}{2+\theta} \cdot n}.$$

$$2 \cdot e^{\frac{-\theta^2}{2+\theta} \cdot n} \leq \varepsilon$$

$$\Rightarrow \quad 2/\varepsilon \leq e^{\frac{\theta^2}{2+\theta} \cdot n}$$

$$\Rightarrow \quad n \geq \left(\frac{2+\theta}{\theta^2}\right) \cdot \ln(2/\varepsilon)$$

Suppose you want $\theta = 0.05$

$$\varepsilon = \frac{1}{100}.$$

$$n \geq 4345.$$

$$p \pm \theta$$

$$[p-\theta, p+\theta]$$

Load Balancing

k servers

n requests

Solution: Send each request to a random server.

$X_1, \ldots, X_k$

X_i : # requests received by server i.

$$p(\max_i X_i \gg \frac{n}{k})?$$

$$X_1 = Y_1 + \cdots + Y_n$$

$$Y_i = \begin{cases} 1 & \text{if request } i \text{ goes to server } 1 \\ 0 & \end{cases}$$

$$p(Y_i = 1) = \frac{1}{k}.$$

$$\mathbb{E}[Y_i] = 1/k.$$

$$\mathbb{E}[X_1] = n/k.$$

$$p\left(X_1 \geq \frac{n}{k} + 3\sqrt{\frac{n \ln k}{k}}\right) \leq \frac{1}{k^3}$$