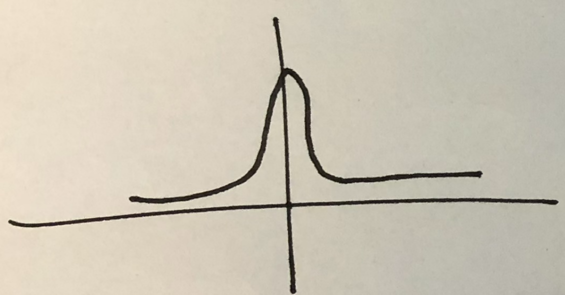
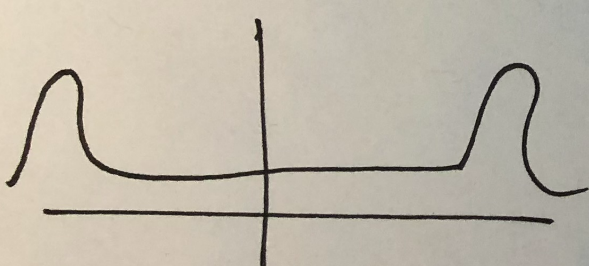
Midterm: Next class!

Review Session: Low 102
2:30 – 4:20
Today.

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mathbb{E}[X])^2]$$





If X, Y are independent

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y).$$

FACT: $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Pf: $\mathbb{E}[X] = \mu.$

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[(X-\mu)^2] \\
&= \mathbb{E}[X^2 + \mu^2 - 2X\cdot\mu] \\
&= \mathbb{E}[X^2] + \mathbb{E}[\mu^2] - 2\mu\cdot\mathbb{E}[X] \\
&= \mathbb{E}[X^2] + \mu^2 - 2\mu^2 \\
&= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2
\end{aligned}$$

Defn: Standard deviation

$$\sigma(X) = \sqrt{Var(X)}.$$

Binomial Distribution

Toss a coin n times. $p(\text{heads}) = p$

X : # of heads.

$$\mathbb{E}[X] = pn \qquad \mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

$$\mathbb{E}[X_i] = p \cdot 1 + (1-p)0$$
$$= p$$

$$\cancel{\mathbb{E}[X^2]} \qquad \mathbb{E}[X_i^2] = p \cdot 1^2 + (1-p) \cdot 0^2$$
$$= p$$

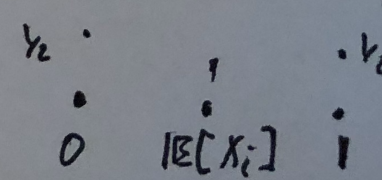
$$Var(X_i) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2$$
$$= p - p^2$$

$$Var(X) = Var(X_1 + X_2 + \cdots + X_n)$$
$$= Var(X_1) + \cdots + Var(X_n)$$
$$= (p - p^2)n$$

$p = 1/2$

$$Var(X_i) = 1/2 - 1/4 = 1/4$$

$$\sigma(X_i) = \sqrt{1/4} = 1/2.$$



$\sigma = 1/2$

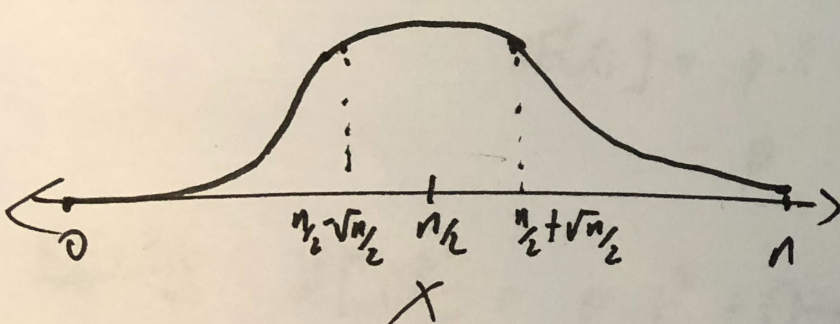
$\mu = 1/2$

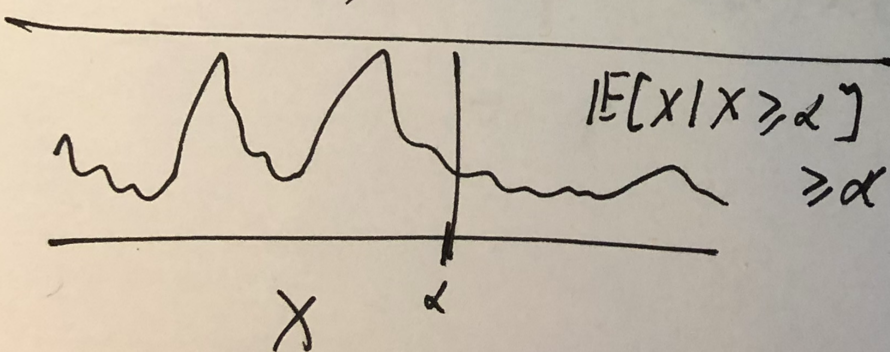
$p(|X_i - \mu| \geq \sigma)$
$= 1$

$$\mathbb{E}[X] = \frac{n}{2}.$$

$$\mathrm{Var}[X] = \frac{n}{4}$$

$$\sigma(X) = \frac{\sqrt{n}}{2}$$





Markov's Inequality

If X is a non-negative random var

$$p(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}.$$

$$\hookrightarrow \quad p(X \geq k \cdot \mathbb{E}(X)) \leq \frac{1}{k}$$

Pf: $$\mathbb{E}[X] = p(X \geq \alpha) \cdot \mathbb{E}[X \mid X \geq \alpha] + p(X < \alpha) \cdot \mathbb{E}[X \mid X < \alpha]$$

$$\geq p(X \geq \alpha) \cdot \alpha$$

$$\Rightarrow \quad p(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}.$$

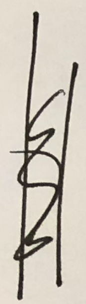
$$\hookrightarrow \quad p(Y \geq \gamma) \leq \frac{\mathbb{E}[Y]}{\gamma} \quad \Bigg| \quad \begin{array}{l} Y = (X - \mathbb{E}[X])^2 \\ \gamma = \alpha^2 \end{array}$$

Chebyshev's Inequality

$$p(|X - \mathbb{E}[X]| \geq \alpha) \leq \frac{\mathrm{Var}[X]}{\alpha^2}$$

$\Leftrightarrow$

$$p(|X-\mathbb{E}[X]| \geq k\sigma(X)) \leq \frac{\mathrm{Var}[X]}{k^2 \cdot (\sigma(X))^2}$$



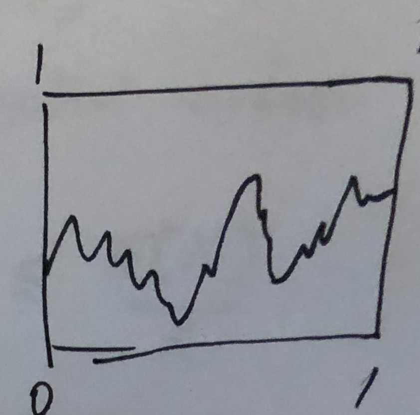
$$= \frac{1}{k^2}$$

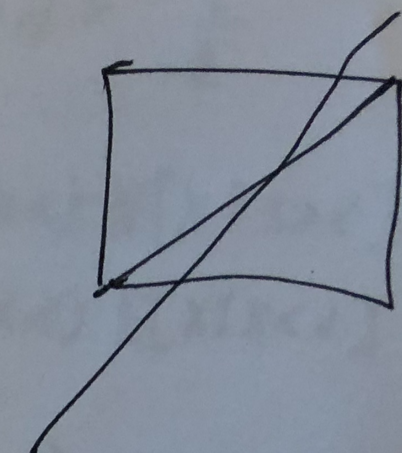
Pf:

$$p((X-\mathbb{E}[X])^2 \geq \alpha^2)$$

$$\leq \frac{\mathbb{E}[(X-\mathbb{E}[X])^2]}{\alpha^2}$$

$$= \frac{\mathrm{Var}(X)}{\alpha^2}$$





Binomial Distribution

$p(\text{heads}) = p$ $\quad$ n coin tosses.

X : # heads

$\mathbb{E}[X] = pn$ $\quad$ | $p \ll 1/2$

$Var(X) = (p - p^2)n$

$p(X \geq 2pn)$

By Markov

$$\leq \frac{\mathbb{E}[X]}{2pn} = \frac{pn}{2pn}$$

$$= \frac{1}{2}.$$

$p(X \geq 2pn)$

$$\leq p(|X - pn| \geq pn)$$

$$\leq \frac{Var(X)}{(pn)^2}$$

$$= \frac{(p-p^2)n}{p^2 \cdot n^2}$$

$$= \frac{(1-p)}{p} \cdot \frac{1}{n}$$