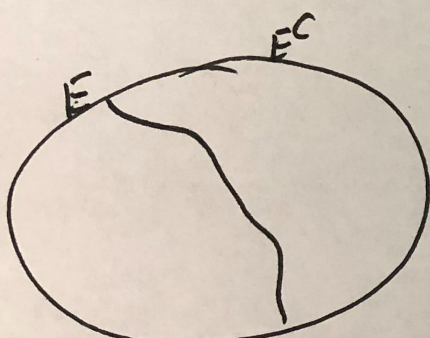
Conditional Expectation

X : r.v.

E : event

$$\mathbb{E}[X \mid E] = \sum_{x \in E} p(X = x \mid E) \cdot x$$

FACT: $\mathbb{E}[X] = p(E) \cdot \mathbb{E}[X \mid E] + p(E^c) \cdot \mathbb{E}[X \mid E^c]$.



$$\mathbb{E}[X] = \sum_x p(X = x) \cdot x$$

$$= \sum_x \left(p(E) \cdot p(X = x \mid E) + p(E^c) \cdot p(X = x \mid E^c) \right) \cdot x$$

$$= \sum_x p(E) \cdot p(X = x \mid E) \cdot x + \sum_x p(E^c) \cdot p(X = x \mid E^c) \cdot x$$

$$= p(E) \cdot \underbrace{\sum_x p(X = x \mid E) \cdot x}_{\mathbb{E}[X \mid E]} + p(E^c) \cdot \underbrace{\sum_x p(X = x \mid E^c) \cdot x}_{\mathbb{E}[X \mid E^c]}$$

$$= p(E) \cdot \mathbb{E}[X \mid E] + p(E^c) \cdot \mathbb{E}[X \mid E^c]$$

Geometric Random Variables

Toss a coin over and over until you see heads. Then stop

$X = \#$ of tosses.

$$\mathbb{E}[X] = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \cdots$$

$$= \sum_{i=1}^{\infty} \frac{i}{2^i} = \left(\frac{1}{2}\right)\cdot\sum_{i=1}^{\infty}\frac{i}{2^{i-1}}$$

$$(1-x^{r+1}) = (1-x)(1+x+x^2+\cdots+x^r)$$

$$\Rightarrow \quad 1+x+\cdots+x^r = \frac{1-x^{r+1}}{1-x} \text{ when } x\neq 1.$$

When $|x|<1$

$$1+x+x^2+\cdots = \frac{1}{1-x} = (1-x)^{-1}$$

Taking derivatives

$$1+2x+3x^2+4x^3+\cdots$$

$$= \frac{1}{(1-x)^2} \quad = (-1)(1-x)^{-2}\cdot(-1)$$

$$\left(\frac{1}{2}\right)\frac{1}{(1-1/2)^2} = \left(\frac{1}{2}\right)\frac{1}{1/4} = \boxed{2}$$

E: event first coin toss is heads

$$\mathbb{E}[X \mid E] = 1$$

$$\mathbb{E}[X \mid E^c] = 1 + \mathbb{E}[X]$$

T????...

??????

When $|x| < 1$

$$1 + x + x^2 \cdot + \quad = \frac{1}{1-x} = (1-x)^{-1}$$

Taking derivatives

$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \frac{1}{(1-x)^2} = (-1)(1-x)^{-2} \cdot (-1)$$

$$\left(\frac{1}{2}\right) \frac{1}{(1-1/2)^2} = \left(\frac{1}{2}\right) \frac{1}{1/4} = \boxed{2}$$

E: event first coin toss is heads

$$\mathbb{E}[X \mid E] = 1$$

$$\mathbb{E}[X \mid E^c] = 1 + \underbrace{\mathbb{E}[X]}$$

T????… ??????

$$\mathbb{E}[X] = p(E) \cdot \mathbb{E}[X \mid E] + p(E^c]\, \mathbb{E}[X \mid E^c]$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + \mathbb{E}[X])$$

$$(1 - 1/2)\, \mathbb{E}[X] = \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \mathbb{E}[X] = 1$$

$$\Rightarrow \mathbb{E}[X] = 2.$$

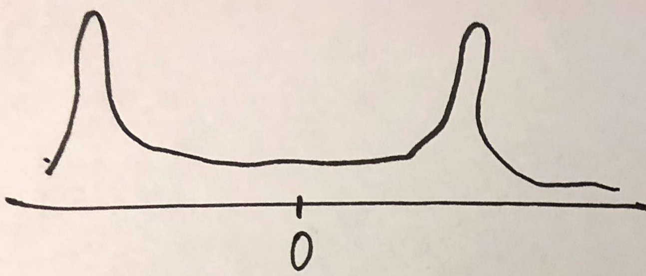
Coin is heads w.p. p.

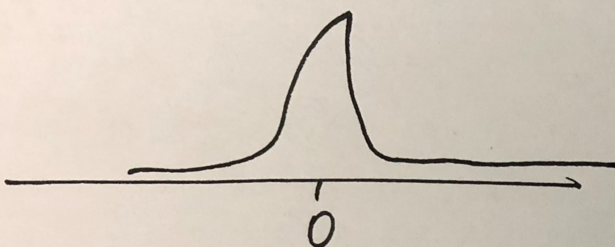
$$\mathbb{E}[X] = 1/p$$

$$\mathbb{E}[X] = p \cdot 1 + (1-p)(1 + \mathbb{E}[X])$$

$$(1 - (1-p))\, \mathbb{E}[X] = p + 1 - p = 1$$

$$\Rightarrow p\,\mathbb{E}[X] = 1 \Rightarrow \mathbb{E}[X] = 1/p.$$





Suppose $\mathbb{E}[X] = 0$.

$\cancel{Var[X] = \mathbb{E}[X^2]}$ $\times$

$Var[X] = \mathbb{E}[(X-\mu)^2]$ where

$\mu = \mathbb{E}[X]$.

$\mathbb{E}[X^k]$: k^{th} moment of X.

a, b

$Var(aX+b)$

$= \mathbb{E}[(aX+b-\mathbb{E}[aX+b])^2]$

$= \mathbb{E}[(aX+b-(a\cdot\mathbb{E}[X]+b))^2$

$= \mathbb{E}[(aX-a\mathbb{E}[X])^2]$

$= \mathbb{E}[a^2(X-\mathbb{E}[X])^2]$

$= a^2 \cdot \mathbb{E}[(X-\mathbb{E}[X])^2]$

$$\boxed{Var(aX+b) = a^2 \cdot Var[X].}$$

$X \qquad Y$

When X, Y are independent

$$Var(X+Y) = var(X) + var(Y)$$

$$X' = X - \mathbb{E}[X]$$

$$Y' = Y - \mathbb{E}[Y]$$

$$Var(X+Y) = Var(X'+Y')$$

$$= \mathbb{E}[(X'+Y')^2]$$

$$= \mathbb{E}[X'^2 + Y'^2 + 2X'Y']$$

$$= \mathbb{E}[X'^2] + \mathbb{E}[Y'^2] + 2\cdot\mathbb{E}[X'Y']$$

$$= Var[X] + Var[Y] + 2\cdot\cancel{\mathbb{E}[X']\cdot\mathbb{E}[Y']}$$

If A, B are indep.

then $\mathbb{E}[A\cdot B] = \mathbb{E}[A]\cdot\mathbb{E}[B]$

$$\sum_{a,b} p(A=a, B=b)\cdot ab$$

$$= \sum_{a,b} p(A=a)\cdot a\cdot\ p(B=b)\cdot b$$

$$= \left(\sum_a p(A=a)\cdot a\right)\left(\sum_b p(B=b)\cdot b\right)$$

$$= \mathbb{E}[A]\cdot\ \mathbb{E}[B].$$