Expected Value

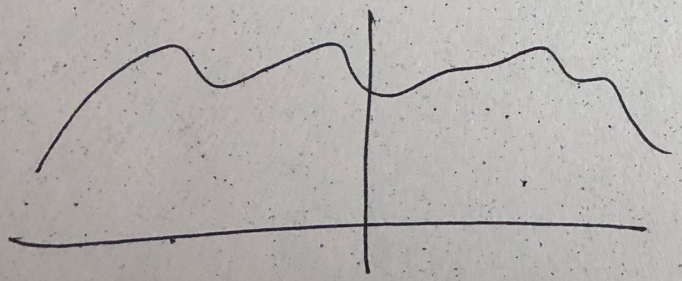
X : real valued r.v.

$$\mathbb{E}[X] = \sum_x p(X=x) \cdot x$$

X: outcome rolling 6-sided die

$$\mathbb{E}[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \cdots + \frac{1}{6} \cdot 6$$

$$= \frac{21}{6} = 3.5$$



X : # number of heads in n coin tosses.

$$\mathbb{E}[X] = \frac{\binom{n}{0}}{2^n} \cdot 0 + \frac{\binom{n}{1}}{2^n} \cdot 1 + \cdots +$$

$$= n/2$$

$$X = \begin{cases} -1000 & \text{w.p } 1/2 \\ 1000 & \text{w.p } 1/2 \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{2}(-1000) + \frac{1}{2}(1000) = 0$$

$$Y = \begin{cases} 1 & \text{w.p } \frac{n-1}{n} \\ -(n-1) & \text{w.p } \frac{1}{n} \end{cases}$$

$$\mathbb{E}[Y] = \frac{n-1}{n} + \frac{1}{n}(-(n-1)) = 0$$

$$Z = 0 \text{ w.p } 1.$$

$$\mathbb{E}[Z] = 0$$

Linearity of Expectation

X, Y

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

works even if X, Y are <u>not</u> independent!

$$\mathbb{E}[XY] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

When X, Y are independent

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

X : # heads in n coin tosses.

$\mathbb{E}[X] = ?$

$X_1 = \begin{cases} 1 & \text{if first coin toss is heads} \\ 0 & \text{o.w.} \end{cases}$

$X_2 =$

$\vdots$

$X_n =$

$$X = X_1 + X_2 + \cdots + X_n$$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_n]$$

$$= \mathbb{E}[X_1] + \mathbb{E}[X_2 + \cdots + X_n]$$

$$= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \cdots + \mathbb{E}[X_n]$$

$$\mathbb{E}[X_1] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

$$= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \frac{n}{2}.$$

Birthday Paradox

107 students

X = # pairs of students with the same birthday.

$$X_{i,j} = \begin{cases} 1 & \text{if } i,j \text{ have same birthday} \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i<j} X_{i,j}\right] = \sum_{i<j} \mathbb{E}[X_{i,j}]$$

$$\mathbb{E}[X_{i,j}] = p(i,j \text{ have same birthday}) \cdot 1 + p(i,j \text{ have diff b.days}) \cdot 0$$

$$= \frac{1}{365} \cdot 1 + 0 = \frac{1}{365}$$

$$\rightarrow \sum_{i<j} \frac{1}{365} = \frac{\binom{107}{2}}{365} = 15.5\ldots$$

lines: d apart
needle: ℓ ← length | $\ell < d$

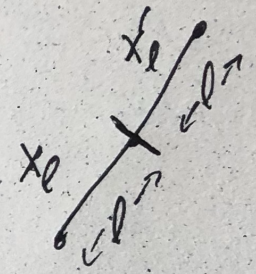
$$p(\text{needle hits line}) = \frac{2\ell}{\pi d} \approx \frac{1}{\pi}$$

ℓ

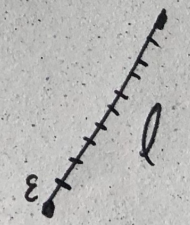
X_ℓ : number of lines hit by needle.

$$\mathbb{E}[X_\ell] = p(0 \text{ line is hit}) \cdot 0 + p(1 \text{ line}) \cdot 1 + p(2 \text{ lines}) \cdot 2 + \dots$$

→ $p(1 \text{ line is hit})$



$$\mathbb{E}[X_{2\ell}] = \mathbb{E}[X_\ell + X'_\ell] = \mathbb{E}[X_\ell] + \mathbb{E}[X'_\ell] = 2 \cdot \mathbb{E}[X_\ell]$$

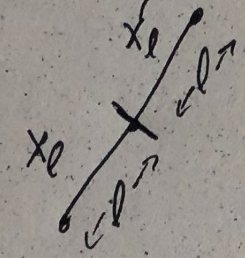


$$\mathbb{E}[X_\ell] = \frac{\ell}{\varepsilon} \cdot \mathbb{E}[X_\varepsilon]$$

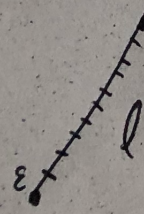
$$\mathbb{E}[X_1] = \frac{1}{\varepsilon} \cdot \mathbb{E}[X_\varepsilon]$$

$$\Rightarrow \boxed{\mathbb{E}[X_\ell] = \ell \cdot \mathbb{E}[X_1]}$$

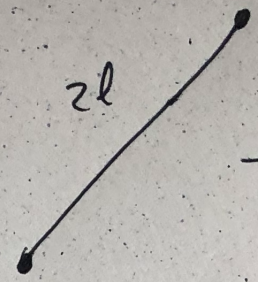
→ p(1 line is hit)

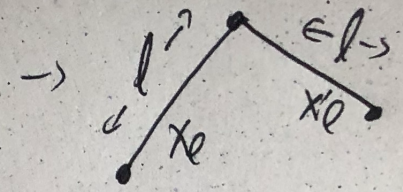


$$\mathbb{E}[X_{2\ell}] = \mathbb{E}[X_\ell + X'_\ell]$$
$$= \mathbb{E}[X_\ell] + \mathbb{E}[X'_\ell]$$
$$= 2 \cdot \mathbb{E}[X_\ell]$$



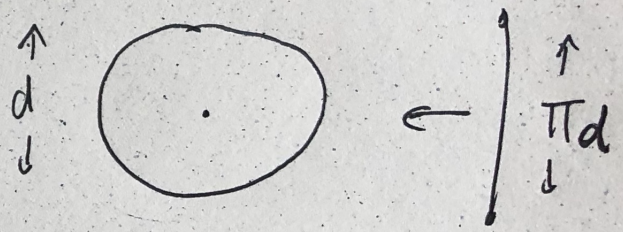
$$\mathbb{E}[X_\ell] = \frac{\ell}{\varepsilon} \cdot \mathbb{E}[X_\varepsilon]$$
$$\mathbb{E}[X_1] = \frac{1}{\varepsilon} \cdot \mathbb{E}[X_\varepsilon]$$
$$\Rightarrow \boxed{\mathbb{E}[X_\ell] = \ell \cdot \mathbb{E}[X_1]}$$



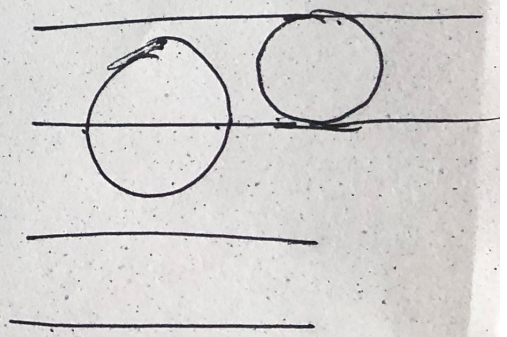


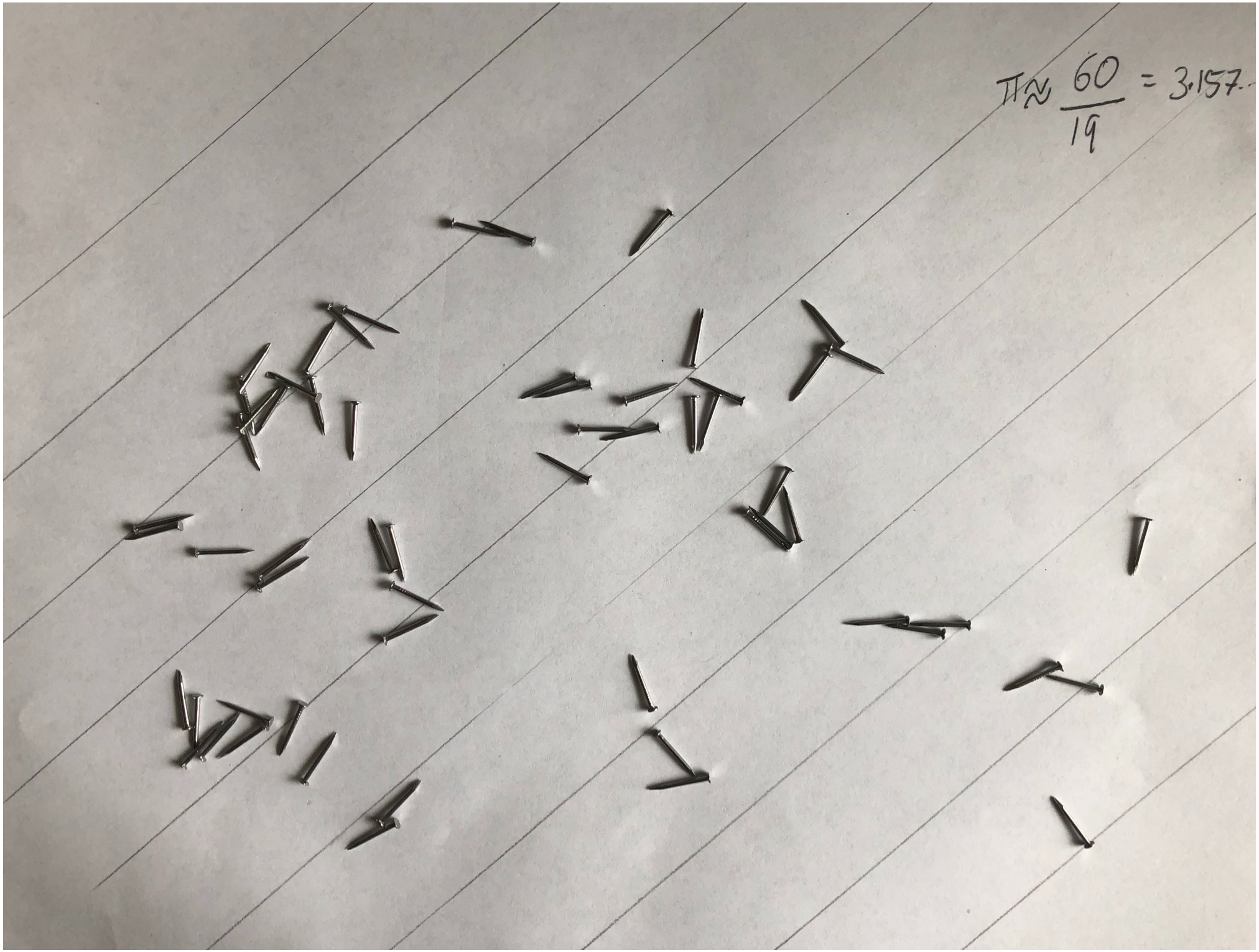
B: # times bent needle hits

$$\mathbb{E}[B] = 2\,\mathbb{E}[X_\ell] = \mathbb{E}[X_{2\ell}]$$



$$\mathbb{E}[X_{\pi d}] = 2$$
$$= \pi d \cdot \mathbb{E}[X_1]$$
$$\Rightarrow \mathbb{E}[X_1] = \frac{2}{\pi d}$$
$$\Rightarrow \mathbb{E}[X_\ell] = \frac{2 \cdot \ell}{\pi d}$$




$\pi \approx \frac{60}{19} = 3.157...$