CSE 312A: Foundations of Computing II
Assignment #7
February 22
due: February 28, at noon.

Instructions:

Answers: When asked for a short answer (such as a single number), also *show and explain your work* briefly. Simplify your final formula algebraically as much as possible, without using your calculator. Then, if the answer is a number rather than a function of some variables, use a calculator to evaluate it and provide the number. For example, for counting problems, your answer might look like this:

Answer: $\binom{5}{2} - \binom{4}{2} = 4$.

Explanation: There are $\binom{5}{2}$ ways to select 2 fingers out of the 5, and $\binom{4}{2}$ of them do not involve the thumb.

Solutions that do not show enough work may not get full credit.

Turn-in: Do not write your name on your pages (your Gradescope account will identify you to us) and do not include a copy of the exercise's question in what you turn in. You must use Gradescope to upload your homework solutions. You will submit a single PDF file containing your solutions to all the exercises in the homework. Each numbered homework question must be answered on its own page (or pages). You must follow the Gradescope prompts that have you link exercise numbers to your pages. You may typeset your solutions on a computer (see here for tutorials and templates) or you can handwrite them, take a picture of (or scan) each handwritten page, and convert the pictures into a single PDF file. You are responsible for making sure that your solution is easily readable, and submitted on time.

CDF of normal Some of the problems require looking up the CDF of the normal distribution. You can use this table to find the values you need.

1. Suppose you have a die that has probability p of resulting in the outcome 6 when rolled, where p is a continuous random variable that is uniformly distributed over $[0, \frac{1}{3}]$. Suppose you start rolling this die. (The value of p does not change once you start rolling.) Give exact answers as simplified fractions.

   (a) Compute the probability that the first roll is 6. (Hint: The answer is not simply "p", since p is a random variable. Think first about *discrete* uniform cases such as the one in which p can take any of the 10 values 1/30, 2/30, ..., 10/30, each with probability 1/10. From here, though, don't take the limiting case as the number of discrete values increases. Instead, use the continuous analog of your discrete case and integrate.) Compare your answer with the probability that the first roll is 6 if it were a fair die.

   (b) Compute the probability that the first two rolls are both 6. (Hint: You cannot assume without proof that the outcomes of two rolls are independent.)

   (c) Let S_1 be the event that the first roll is 6 and S_2 be the event that the second roll is 6. Compute $P(S_2 \mid S_1)$.

(d) Are the outcomes of the first two rolls independent? Justify your answer.

(e) Compute the probability that the first k rolls are each 6. Compare this with the probability if it were a fair die.

2. Suppose X and Y are independent, and all are uniformly distributed real numbers between 0 and 1.

   (a) Derive the pdf of $X + Y$.

   (b) Derive the pdf of X^3.

   (c) Derive the pdf of $X^3 + Y$.

3. I win 22.5% of chess games played against a chess simulator, independently of each other. Use the Central Limit Theorem to estimate the probability that, in a random sample of 100 future games I play against the simulator:

   (a) I will win more than 23.

   (b) I will win between 22 and 29 (inclusive).

4. Let X be the sum of 3,000,000 real numbers, and Y be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and each one is uniformly distributed over $(-0.5, +0.5)$, use the Central Limit Theorem to estimate the probability that $|X - Y| > 1200$. Noticing that $|X - Y|$ could have been as great as 1,500,000, look at your answer and think about what it says.

5. Suppose packet arrivals at a particular server follow a Poisson distribution, with an average rate of 8 packet arrivals per millisecond. Use the Central Limit Theorem to approximate the probability that, in 100 milliseconds, at least 775 packets will arrive at this server. You may assume packet arrivals are independent of each other.

6. Suppose you are conducting an experiment and want to estimate the variance of a variable X. You know that X always lies in between 0 and 10, it has expectation 5, $p(X < 2) \leq 1/3$, and $p(X > 8) \leq 1/3$. Given these constraints, what is the distribution for X that has the maximum variance? Describe the distribution, and show that no other distribution can have a larger variance. Hint: You can shift X and the variance does not change, so start by picking a convenient shift. Then write down an expression for the variance, breaking up the integral into different regions. Bound the contribution of each region separately.