CSE 312A: Foundations of Computing II
Assignment #5
February 8
due: February 14, at noon.

**Instructions:**

- **Answers:** When asked for a short answer (such as a single number), also *show and explain your work* briefly. Simplify your final formula algebraically as much as possible, without using your calculator. Then, if the answer is a number rather than a function of some variables, use a calculator to evaluate it and provide the number. For example, for counting problems, your answer might look like this:

  Answer: $\binom{5}{2} - \binom{4}{2} = 4$.

  Explanation: There are $\binom{5}{2}$ ways to select 2 fingers out of the 5, and $\binom{4}{2}$ of them do not involve the thumb.

  Solutions that do not show enough work may not get full credit.

- **Turn-in:** Do not write your name on your pages (your Gradescope account will identify you to us) and do not include a copy of the exercise's question in what you turn in. You must use Gradescope to upload your homework solutions. You will submit a single PDF file containing your solutions to all the exercises in the homework. Each numbered homework question must be answered on its own page (or pages). You must follow the Gradescope prompts that have you link exercise numbers to your pages. You may typeset your solutions on a computer (see here for tutorials and templates) or you can handwrite them, take a picture of (or scan) each handwritten page, and convert the pictures into a single PDF file. You are responsible for making sure that your solution is easily readable, and submitted on time.

1. Suppose $X$ is a random variable with $p(X = 0) = 4/5$, $p(X = 1) = 1/10$, $p(X = 9) = 1/10$. Then
   (a) Compute $\mathrm{Var}[X]$ and $\mathbb{E}[X]$.
   (b) What is the upper bound on the probability that $X$ is at least 9 obained by applying Markov's inequality?
   (c) What is the upper bound on the probability that $X$ is at least 9 obained by applying Chebychev's inequality?
   (d) What is the upper bound on the probability that $X$ is at least 20 obained by applying Markov's inequality?
   (e) What is the upper bound on the probability that $X$ is at least 20 obained by applying Chebychev's inequality?

2. You are playing a game that uses a fair 12-sided die whose faces are numbered 1, 2, $\ldots$, 12. The value of a roll is the number showing on the top of the die when it comes to rest. Give all answers as simplified fractions.
   (a) Let $X$ be the value of one roll of the die. Computer $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.
   (b) Let $Y$ be the sum of the values of 4 independent rolls of the die. Compute $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$. Use independence, and state precisely where in your computation you are using it.

(c) Let $Z$ be the average of the values of 4 independent rolls of the die. Compute $\mathbb{E}[Z]$ and $\mathrm{Var}[Z]$. Use independence, and state precisely where in your computation you are using it.

3. Let X be a random variable with expected value $\mu$ and variance $\sigma^2$. Find the expected value and variance of $Y = (X - \mu)/\sigma$. (Your answer will help explain why $Y$ is called the "standardized" version of $X$: standardizing two random variables puts them on more of an equal footing for comparing, as your answer will show.)

4. In class, we analyzed Buffon's needle experiment. We showed that if a large sheet of paper has parallel lines that are 1 inch apart, and we throw a needle of length $1/2$ inch at it, the probability that the needle hits a line is $1/\pi$. We tried to estimate $\pi$ by performing an the experiment of throwing many nails a few times and seeing how many throws hit a line. Our throws were not really independent, because the nails collided with each other. We could have made them independent by throwing them one at a time, but this would be much more time consuming.

   Suppose we throw a needle $n$ times, and each throw is independent. Let $X$ be the number of throws where the needle hits a line. Our estimate for $\pi$ is $Y = n/X$. Let us figure out how many times we need to throw the needle to get a reasonably accurate estimate for $\pi$. (Later in the class, we shall prove much stronger concentration bounds that will show that much fewer samples are necessary to get a good estimate).

   (a) Calculate $\mathbb{E}[X]$.

   (b) Calculate $\mathrm{Var}[X]$.

   (c) Calculate $\mathrm{Var}[1/Y]$.

   (d) If $|\alpha - \beta| \le \epsilon$, with $\alpha, \beta > 0$ and $\epsilon \ll \alpha, \beta$, then $|1/\alpha - 1/\beta| = \left|\frac{\beta-\alpha}{\alpha\beta}\right| \le \frac{\epsilon}{\alpha(\alpha-\epsilon)}$. In our case, setting $\alpha = 1/\pi$ and $\beta = 1/Y$, and $\epsilon = 1/1000$, we have

   $$\frac{\epsilon}{\alpha(\alpha - \epsilon)} = \frac{1/1000}{(1/\pi)(1/\pi - 1/1000)} \le \frac{4^2}{1000} = 0.016,$$

   just by using the crude estimate that $3 \le \pi \le 4$. So, we get that

   $$p(|Y - \pi| \ge 0.016) \le p(|1/Y - 1/\pi| \ge 1/1000).$$

   Compute how large $n$ needs to be so that we can use Chebyshev's inequality to prove that

   $$p(|1/Y - 1/\pi| \ge 1/1000) \le 1/100.$$